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TOP SCHOLAR



Mana Tohu Mātauranga o Aotearoa
New Zealand Qualifications Authority

Scholarship 2024 Calculus

Time allowed: Three hours
Total score: 32

Check that the National Student Number (NSN) on your admission slip is the same as the number at the top of this page.

You should answer ALL the questions in this booklet.

Pull out Formulae and Tables Booklet S–CALCF from the centre of this booklet.

Show ALL working. Correct answers only will not be sufficient.

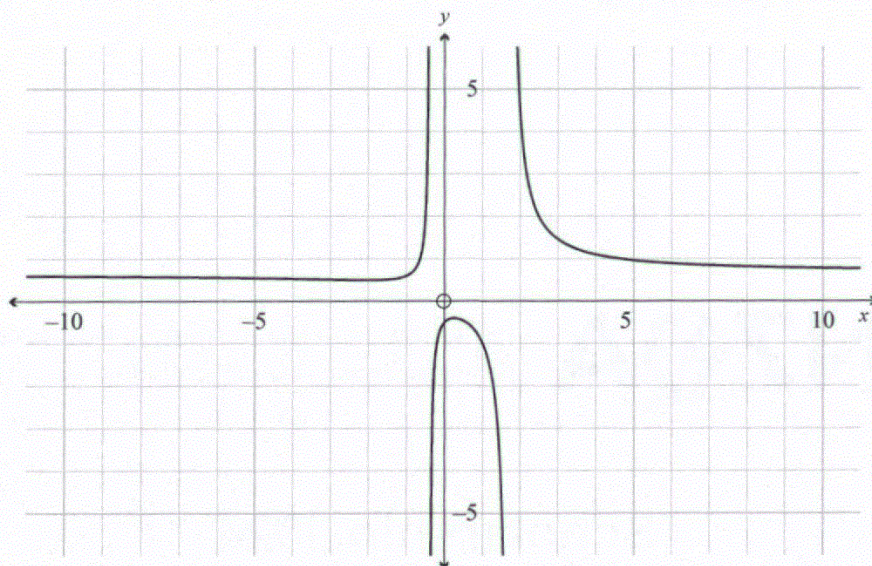
Check that this booklet has pages 2–24 in the correct order and that none of these pages is blank.

Do not write in any cross-hatched area (///). This area will be cut off when the booklet is marked.

YOU MUST HAND THIS BOOKLET TO THE SUPERVISOR AT THE END OF THE EXAMINATION.

QUESTION ONE

- (a) Consider the curve $y = \frac{2x^2+1}{3x^2-4x-2}$, which is shown below.



- (i) Find the coordinates of any stationary points on the curve, and determine their nature.

$$y = \frac{2x^2+1}{3x^2-4x-2}$$

$$\frac{dy}{dx} = \frac{f'g - g'f}{g^2} = \frac{4x(3x^2-4x-2) - (6x-4)(2x^2+1)}{(3x^2-4x-2)^2} = 0$$

$$12x^3 - 16x^2 - 8x - (12x^3 - 8x^2 + 6x - 4) = 0$$

$$-8x^2 - 14x + 4 = 0$$

$$4x^2 + 7x - 2 = 0$$

$$(4x-1)(x+2) = 0$$

$$\begin{cases} x = \frac{1}{4} & \text{or} & x = -2 \\ \hookrightarrow y = -\frac{2}{5} & & \hookrightarrow y = \frac{1}{2} \end{cases}$$

We use FDT to classify their nature:

$\frac{dy}{dx}$	$x=0.2$	$x=0.25$	$x=0.3$	$\frac{dy}{dx}$	$x=-2.1$	$x=-2$	$x=-1.9$
	0.123	0	-0.107		-4.88×10^{-3}	0	6.37×10^{-3}
		$\nearrow$	$\searrow$		$\searrow$	$\nearrow$	
		$\therefore$ maximum			$\therefore$ minimum		

Therefore the stationary points are: $\left(\frac{1}{4}, -\frac{2}{5}\right)$ maximum
 $\left(-2, \frac{1}{2}\right)$ minimum

- (ii) Find the value of $\lim_{x \rightarrow \infty} \frac{2x^2+1}{3x^2-4x-2}$.

Hence, find the coordinate(s) where the curve intersects its own asymptote(s).

$$\begin{aligned} \lim_{x \rightarrow \infty} \left[\frac{2x^2+1}{3x^2-4x-2} \right] &= \lim_{x \rightarrow \infty} \left[\frac{2 + \frac{1}{x^2}}{3 - \frac{4}{x} - \frac{2}{x^2}} \right] \\ &= \frac{2+0}{3-0-0} = \frac{2}{3}. \end{aligned}$$

Note also that $\lim_{x \rightarrow -\infty} \left[\frac{2x^2+1}{3x^2-4x-2} \right] = \frac{2}{3}$ by same reasoning.

$\therefore$ horizontal asymptote is $y = \frac{2}{3}$.

Vertical asymptotes arise when denominator is zero:

$$3x^2-4x-2=0, \quad x = \frac{2 \pm \sqrt{2^2 - (3)(-2)}}{3} = \frac{2 \pm \sqrt{10}}{3}$$

However, the curve will not intersect its vertical asymptotes because the curve is undefined on the x -values of the vertical asymptotes (due to division by zero).

Hence we need only find the intersection(s) with the horizontal asymptote $y = \frac{2}{3}$.

$$y = \frac{2}{3} = \frac{2x^2+1}{3x^2-4x-2}$$

$$2(3x^2-4x-2) = 3(2x^2+1)$$

$$6x^2-8x-4 = 6x^2+3$$

$$x = -\frac{7}{8}$$

$\therefore$ only curve $\leftrightarrow$ asymptote intersection is the point $(x, y) = \left(-\frac{7}{8}, \frac{2}{3}\right)$.

(b) For what real value(s) of α does the following system of equations have no real solutions?

$$x^3 + y^3 = 2$$

$$x + y = \alpha$$

$$x^3 + y^3 = (x+y)(x^2 - xy + y^2) = 2$$

$$\alpha (x^2 - xy + y^2) = 2$$

$$\alpha (x^2 + 2xy + y^2 - 3xy) = 2$$

$$\alpha ((x+y)^2 - 3xy) = 2$$

$$\alpha (\alpha^2 - 3xy) = 2$$

Case #1. $\alpha = 0$.

Then ~~0~~ $0 = 2$, contradiction, so no real solutions.

~~Case #2. $\alpha \neq 0$.~~

~~Then $\alpha^2 - 3xy = \frac{2}{\alpha}$~~

~~$\Rightarrow 3xy = \alpha^2 - \frac{2}{\alpha}$~~

~~$\Rightarrow xy = \frac{1}{3} \left[\alpha^2 - \frac{2}{\alpha} \right]$~~

~~Now $x+y = \alpha$ and $xy = \frac{1}{3} \left[\alpha^2 - \frac{2}{\alpha} \right]$, so x and y are~~

Case #2. $\alpha \neq 0$, then $\alpha^2 - 3xy = \frac{2}{\alpha}$

$$\Rightarrow xy = \frac{1}{3} \left[\alpha^2 - \frac{2}{\alpha} \right]$$

Now $x+y = \alpha$ and $xy = \frac{1}{3} \left[\alpha^2 - \frac{2}{\alpha} \right]$,

so x and y are the roots of the equation

$$t^2 - \alpha t + \frac{1}{3} \left[\alpha^2 - \frac{2}{\alpha} \right] = 0 \quad \left(\begin{array}{l} \text{as per sum and} \\ \text{products of roots.} \end{array} \right)$$

$$\Delta = \alpha^2 - 4(1) \left(\frac{1}{3} \left[\alpha^2 - \frac{2}{\alpha} \right] \right) < 0 \text{ for no real solutions}$$

$$3\alpha^2 - 4 \left(\alpha^2 - \frac{2}{\alpha} \right) < 0$$

$$-\alpha^2 + \frac{8}{\alpha} < 0$$

$$\frac{8}{\alpha} < \alpha^2$$

If $\alpha < 0$ then this is true as $\alpha^2 > 0 > \frac{8}{\alpha}$

If $\alpha > 0$ then $\frac{8}{\alpha} < \alpha^2 \Leftrightarrow \alpha^3 > 8 \Leftrightarrow \alpha > 2$.

So, in conclusion, the real values of α for which there are no solutions are $\alpha \leq 0$ and $\alpha > 2$, i.e.

- (c) A landscaper is building steps up one side of a 48 metre high hill.

The slope of this side of the hill can be modelled by the curve

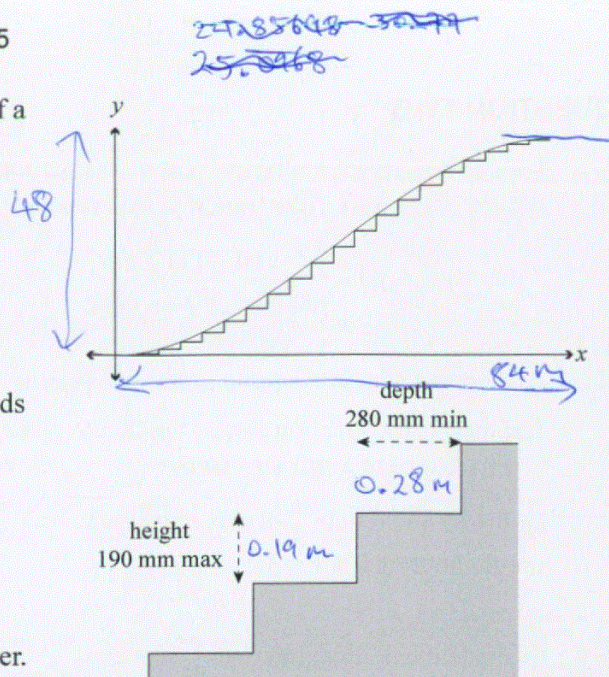
$$y = kx^2(126 - x)$$

where both x and y are in metres and k is a constant to be determined.

Building regulations state that each step needs to have a **minimum** depth of 280 mm and a **maximum** height of 190 mm, as shown.

If the landscaper chooses to build each step with a depth of 280 mm, would the steps all satisfy the maximum height regulations?

You must use calculus to support your answer.



Presume that at $y=48$, the curve has a local maximum / gradient of zero (because hills are not jagged! $\leftarrow$ not this)
 $y = k(126x^2 - x^3) \Rightarrow \frac{dy}{dx} = k(252x - 3x^2) = 0$
 rejecting $x=0$, $252 - 3x = 0 \Rightarrow x = 84$. So the hill is 84 m in the x -direction, and $48 = k(84)^2(126 - 84) \Rightarrow k = \frac{1}{6174}$.

Let the step height going from x to $x + 0.28$ be Δy ,
 i.e. $\Delta y = f(x + 0.28) - f(x)$

$$= \frac{1}{6174} [126((x + 0.28)^2 - x^2) - ((x + 0.28)^3 - x^3)]$$

$$= \frac{1}{6174} [126(0.56x + 0.0784) - (3x^2(0.28) + 3x(0.28)^2 + 0.28^3)]$$

$$= \frac{1}{6174} [126(2\lambda x + \lambda^2) - (3x^2\lambda + 3x\lambda^2 + \lambda^3)]$$

 (where $\lambda = 0.28$)

$$= \frac{1}{6174} [-3\lambda x^2 + (252\lambda - 3\lambda^2)x + 126\lambda^2 - \lambda^3].$$

Now for a curve $y = ax^2 + bx + c$ the vertex is found at $y' = 0$,
 i.e. $2ax + b = 0$ i.e. $x = -\frac{b}{2a}$. Therefore, the maximum Δy
 is when $x = \frac{252\lambda - 3\lambda^2}{-2(-3\lambda)} = 41.86$, and at this x , we have
 $(\Delta y)_{\max} = 0.2399... \approx 240$ mm. Although 41.86 is not
 an exact multiple of 0.28, 41.72 and 42 are, and the step-size
 between these x -values is also ≈ 240 mm. Therefore, no,
 the steps do not satisfy regulations, as 240 mm > 190 mm.

QUESTION TWO

- (a) For any two functions f and g defined on an interval, $a \leq x \leq b$, the **max value** function is defined as:

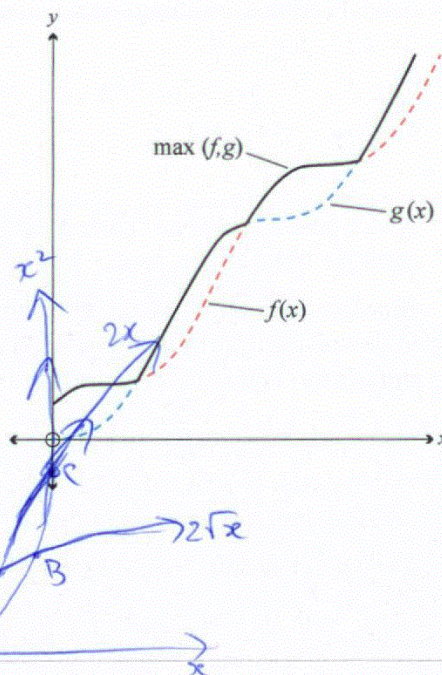
$$\max(f, g) = \begin{cases} f(x) & \text{if } f(x) \geq g(x) \\ g(x) & \text{if } f(x) < g(x) \end{cases}$$

To demonstrate, an example is shown opposite.

The max value function can be applied to more than two functions in a similar manner.

Consider the function $h(x) = \max(2\sqrt{x}, 2x, x^2)$ on the interval $0 \leq x \leq 3$.

Evaluate the definite integral $\int_0^3 h(x) dx$.



A: $2\sqrt{x} = 2x \Leftrightarrow x=1$ (reject $x=0$)
 B: $2\sqrt{x} = x^2 \Leftrightarrow 4x = x^4 \Leftrightarrow x = \sqrt[3]{4}$ (reject $x=0$)
 C: $2x = x^2 \Leftrightarrow x=2$ (reject $x=0$).

Thus, $\int_0^3 h(x) dx = \int_0^1 2\sqrt{x} dx + \int_1^{\sqrt[3]{4}} 2x dx + \int_{\sqrt[3]{4}}^3 x^2 dx$

$$= \left[\frac{2x^{3/2}}{3/2} \right]_0^1 + \left[x^2 \right]_1^{\sqrt[3]{4}} + \left[\frac{x^3}{3} \right]_{\sqrt[3]{4}}^3$$

$$= \frac{4}{3} + (\sqrt[3]{4})^2 - 1 + 9 - \frac{4}{3}$$

$$= \frac{4}{3} + 3\sqrt[3]{8} - 1 + 9 - \frac{4}{3}$$

$$= 2\sqrt[3]{2} + 8$$

- (b) Given that $\frac{dy}{dx} = 1 + y^2$, find the value of M for which $\frac{d^3y}{dx^3} = M(1 + y^2)(1 + 3y^2)$.

$$\frac{dy}{dx} = 1 + y^2$$

$$\frac{d^2y}{dx^2} = 2y \frac{dy}{dx} = 2y(1 + y^2) = 2y + 2y^3$$

$$\begin{aligned} \frac{d^3y}{dx^3} &= 2 \frac{dy}{dx} + 6y^2 \cdot \frac{dy}{dx} \\ &= 2(1 + 3y^2) \frac{dy}{dx} \\ &= 2(1 + 3y^2)(1 + y^2). \end{aligned}$$

$$\therefore \underline{\underline{M = 2}}$$

- (c) For any real number x , the **absolute value** of x is defined as:

$$|x| = \begin{cases} -x & \text{if } x < 0 \\ x & \text{if } x \geq 0 \end{cases}$$

note: $\frac{d}{dx} [|x|] = \frac{|x|}{x}$,

because

$$\frac{|x|}{x} = 1 \text{ when } x > 0$$

$$\text{and } \frac{|x|}{x} = -1 \text{ when } x < 0.$$

Consider the function $s_n(x) = |x-1| + |x-2| + \dots + |x-n|$ with $n > 2$.

- (i) Find $s_3'\left(\frac{5}{2}\right)$, the derivative of $s_3(x)$ evaluated at $x = \frac{5}{2}$.

$$s_3(x) = |x-1| + |x-2| + |x-3|$$

$$s_3'(x) = \frac{|x-1|}{x-1} + \frac{|x-2|}{x-2} + \frac{|x-3|}{x-3}$$

$$s_3'\left(\frac{5}{2}\right) = \frac{|\frac{5}{2}-1|}{\frac{5}{2}-1} + \frac{|\frac{5}{2}-2|}{\frac{5}{2}-2} + \frac{|\frac{5}{2}-3|}{\frac{5}{2}-3} = 1 + 1 + (-1) = \underline{\underline{1}}.$$

- (ii) Find the minimum value of $s_{2024}(x)$.

Justify your answer.

$$s_{2024}(x) = |x-1| + |x-2| + |x-3| + \dots + |x-2024|$$

$$s_{2024}'(x) = \frac{|x-1|}{x-1} + \frac{|x-2|}{x-2} + \frac{|x-3|}{x-3} + \dots + \frac{|x-2024|}{x-2024}$$

(note $s_{2024}'(x)$ not defined on integer x -values due to "jagged" nature of the curve)

I claim the minimum value occurs when $1012 \leq x \leq 1013$.
To prove this first note that:

• if $x < 1012$, then $s_{2024}'(x) = 1 + 1 + \dots + \dots + (-1) + (-1) \dots$

which is negative, so

$s_{2024}(x)$ is strictly decreasing

when $x < 1012$. (integer values of x do not change this)

fewer 1's than (-1)'s $\rightarrow$ more (-1)'s than 1's
negative

• if $x > 1013$, then $s_{2024}'(x) = 1 + 1 + \dots + \dots + (-1) + (-1) \dots$

which is positive, so

$s_{2024}(x)$ is strictly increasing

when $x > 1013$. (Since curve is continuous, the jagged points at integer x values do not change this)

more 1's than (-1)'s $\rightarrow$ less (-1)'s than 1's

• if $1012 < x < 1013$, then $s_{2024}'(x) = 1 + 1 + \dots + \dots + (-1) + (-1) \dots$

which is equal to zero,

so $s_{2024}(x)$ is constant over $1012 < x < 1013$,
and in fact also over $1012 \leq x \leq 1013$.

exactly 1012 1's exactly 1012 (-1)'s

$\rightarrow$ So graphically, the min value occurs over the entire interval $[1012, 1013]$.

QUESTION THREE

- (a) Find the exact value of $\int_0^{\pi/4} \frac{\cos 2x}{\cos^2 x} dx$.

$$\begin{aligned}\int_0^{\pi/4} \frac{\cos 2x}{\cos^2 x} dx &= \int_0^{\pi/4} \frac{2\cos^2 x - 1}{\cos^2 x} dx \\&= \int_0^{\pi/4} 2 - \sec^2 x \, dx \\&= \left[2x - \tan x \right]_0^{\pi/4} \\&= \left(2\left(\frac{\pi}{4}\right) - \tan \frac{\pi}{4} \right) - 0 \\&= \frac{\pi}{2} - 1\end{aligned}$$

1.5
2.5x8

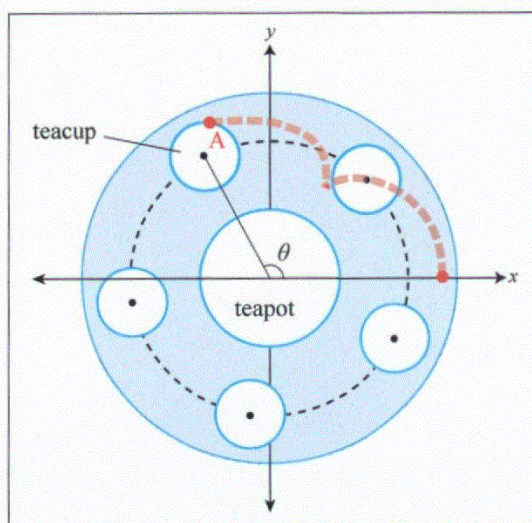
- (b) Alice takes a ride at an amusement park that involves
- five spinning teacups that rotate on circular saucers
 - a circular platform that rotates around a large teapot at its centre.

The coordinates of Alice's motion in the xy -plane (shown as point A in the diagram below) can be described by the parametric equations:

$$x = 4\cos\theta + \cos 4\theta$$

$$y = 4\sin\theta + \sin 4\theta$$

where θ is the anti-clockwise angle shown.



<https://www.flickr.com/photos/nycdc/4691066745>

- (i) Find the coordinates in the first quadrant where Alice is closest to the teapot.
Justify your answer.

$$\begin{aligned}
 r^2 &= \text{Distance squared} = x^2 + y^2 \\
 &= (4\cos\theta + \cos 4\theta)^2 + (4\sin\theta + \sin 4\theta)^2 \\
 &= 16(\cos^2\theta + \sin^2\theta) + 8(\cos\theta \cos 4\theta + \sin\theta \sin 4\theta) + (\cos^2 4\theta + \sin^2 4\theta) \\
 &= 16(1) + 8\cos(4\theta - \theta) + 1 \\
 &= 17 + 8\cos 3\theta.
 \end{aligned}$$

as $\cos x \geq -1 \forall x \in \mathbb{R}$

The minimum value of r^2 is clearly when $\cos 3\theta = -1$ (and thus r)
 The only value of θ , $0^\circ < \theta < 90^\circ$ (first quadrant), for which $\cos 3\theta = -1$ is $3\theta = 180^\circ$ i.e. $\theta = 60^\circ$.

The desired coordinates are thus:

$$\begin{aligned}
 (x, y)_{\min} &= (4\cos 60^\circ + \cos(4 \times 60^\circ), 4\sin 60^\circ + \sin(4 \times 60^\circ)) \\
 &= (4\cos 60^\circ + \cos(180^\circ + 60^\circ), 4\sin 60^\circ + \sin(180^\circ + 60^\circ)) \\
 &= \left(4 \cdot \frac{1}{2} - \frac{1}{2}, 4 \cdot \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}\right) = \left(\frac{3}{2}, \frac{3\sqrt{3}}{2}\right)
 \end{aligned}$$

$$0^\circ \leq \theta \leq 360^\circ$$

- (ii) Find the distance Alice travels in one complete rotation around the teapot.

Note: when a curve is defined parametrically by the equations $x = f(\theta)$ and $y = g(\theta)$ on an interval $\alpha \leq \theta \leq \beta$, we can find its length, L , by using the formula:

$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} d\theta$$

$$\frac{dx}{d\theta} = -4\sin\theta - 4\sin(4\theta)$$

$$\frac{dy}{d\theta} = 4\cos\theta + 4\cos(4\theta)$$

$$L = \int_0^{2\pi} \sqrt{(-4\sin\theta - 4\sin(4\theta))^2 + (4\cos\theta + 4\cos(4\theta))^2} d\theta$$

$$= \int_0^{2\pi} 4 \sqrt{\sin^2\theta + 2\sin\theta\sin(4\theta) + \sin^2(4\theta) + \cos^2\theta + 2\cos\theta\cos(4\theta) + \cos^2(4\theta)} d\theta$$

$$= \int_0^{2\pi} 4 \sqrt{1 + 1 + 2(\sin\theta\sin(4\theta) + \cos\theta\cos(4\theta))} d\theta$$

$$= \int_0^{2\pi} 4 \sqrt{2 + 2\cos(4\theta - \theta)} d\theta$$

$$= \int_0^{2\pi} 4 \sqrt{2 + 2\cos 3\theta} d\theta$$

$$= \int_0^{2\pi} 4 \cdot \sqrt{4 \cdot \frac{1 + \cos 3\theta}{2}} d\theta$$

$$= 8 \int_0^{2\pi} \sqrt{\frac{1 + \cos 3\theta}{2}} d\theta$$

$$= 8 \int_0^{2\pi} \sqrt{\frac{1 + 2\cos^2(\frac{3\theta}{2}) - 1}{2}} d\theta$$

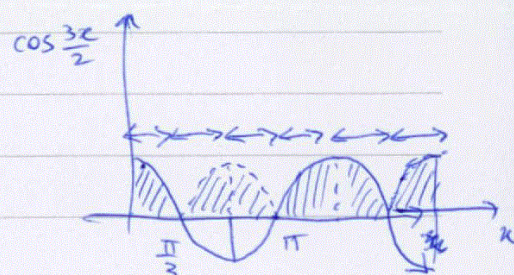
$$= 8 \int_0^{2\pi} \sqrt{\cos^2(\frac{3\theta}{2})} d\theta = 8 \int_0^{2\pi} \left| \cos \frac{3\theta}{2} \right| d\theta$$

$$= 8 \times 6 \times \int_0^{\pi/3} \cos \frac{3\theta}{2} d\theta \rightarrow \text{see diagram}$$

$$= 8 \times 6 \times \left[\frac{2}{3} \sin \frac{3\theta}{2} \right]_0^{\pi/3}$$

$$= 8 \times 6 \times \frac{2}{3} \sin\left(\frac{3 \cdot \frac{\pi}{3}}{2}\right)$$

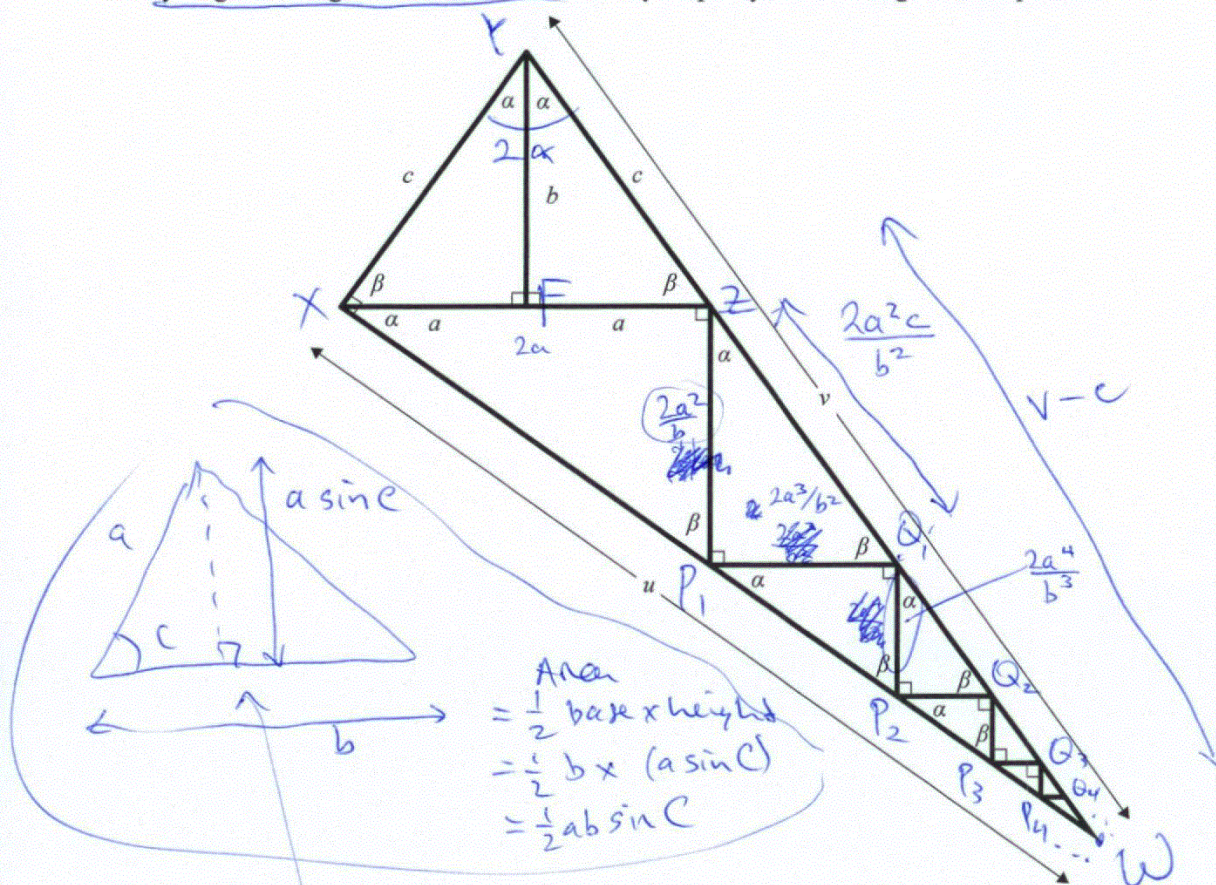
$$= 8 \times 6 \times \frac{2}{3} = \underline{\underline{32 \text{ units}}}$$



- (c) In 2023, two American high-school students discovered a new proof of the Pythagorean Theorem using trigonometry. In this question we will work through the key steps to derive their result.

The students' proof makes use of the diagram below, which consists of an infinite number of **similar** right-angled triangles enclosed within a large right-angled triangle. It was referred to by the students as a "waffle cone".

Note: to avoid circular logic, you should not make use of the Pythagorean Theorem or any of the Pythagorean trigonometric identities at any step in your working for this question.



- (i) Show that $c^2 = \frac{2ab}{\sin 2\alpha}$.

The "Area = $\frac{1}{2}ab \sin C$ " rule does not require Pythagoras in its proof, so we may utilise it here.

Consider the triangle marked XYZ.

$$\text{Area [XYZ]} = \frac{1}{2} (c)(c) \sin(2\alpha) = \frac{1}{2} c^2 \sin(2\alpha)$$

$$\text{Area [XYZ]} = \frac{1}{2} \cdot \text{base} \cdot \text{height} = \frac{1}{2} (2a)(b) = ab.$$

$$\therefore \frac{1}{2} c^2 \sin(2\alpha) = ab$$

$$\Rightarrow c^2 = \frac{2ab}{\sin(2\alpha)}, \text{ as required.}$$

- (ii) Show that $u = \frac{2abc}{b^2 - a^2}$ and find a similar expression for v .

- Total area
- Scale factor

Hence, prove that $a^2 + b^2 = c^2$.

Hint: the following formulae will prove useful:

$$T_n = T_1 r^{n-1} \quad S_n = T_1 \left(\frac{1-r^n}{1-r} \right) \quad \text{scale factor} = \frac{2a}{b}$$

Notice that by similar triangles XZP_1 and YFX , we have $XP_1 = \frac{2ac}{b}$ and $ZP_1 = \frac{2a^2}{b}$.

Now notice that there is a scale factor of $\frac{a}{b}$ going from triangle XZP_1 to similar triangle ZP_1Q_1 (because the "a" side ~~is affected~~ then acts as a "b" side).

Furthermore there is ^{similarly} a scale factor of $\frac{a}{b}$ going from triangle ZP_1Q_1 to similar triangle $P_1Q_1P_2$.

In other words, there is a scale factor of $\frac{a}{b}$ going from some triangle in the infinite waffle cone (starting from XZP_1) to the next triangle in the chain / next smallest.

Since two scale factors of $\frac{a}{b}$ are applied in going from $\triangle XZP_1$ to $\triangle ZP_1Q_1$ to $\triangle P_1Q_1P_2$, we have

~~$XP_1 = \frac{2ac}{b}$~~ $P_1P_2 = \left(\frac{a}{b}\right)\left(\frac{a}{b}\right) \cdot XP_1 = \frac{a^2}{b^2} XP_1$, and then similarly $P_2P_3 = \frac{a^2}{b^2} P_1P_2$, etc. Thus u can be evaluated as the ~~infinite~~ ^{the} sum of terms in the geometric sequence with $T_1 = XP_1 = \frac{2ac}{b}$ and $r = \frac{a^2}{b^2}$.

$$\begin{aligned} \therefore u &= \frac{2ac}{b} \left(\frac{1-r^\infty}{1-r} \right) = \frac{2ac}{b} \left(\frac{1}{1-r} \right) = \frac{2ac}{b} \left(\frac{1}{1-\frac{a^2}{b^2}} \right) \times \frac{b^2}{b^2} \\ &= \frac{2ab^2c}{b(b^2-a^2)} = \frac{2abc}{b^2-a^2}, \text{ as required.} \end{aligned}$$

Similarly we have: $ZQ_1 = \frac{2a^2}{b} \times \frac{c}{b} = \frac{2a^2c}{b^2}$, and so

$$v - c = \frac{2a^2c}{b^2} \left(\frac{1-r^\infty}{1-r} \right) = \frac{2a^2c}{b^2} \left(\frac{1}{1-\frac{a^2}{b^2}} \right) = \frac{2a^2c}{b^2-a^2}$$

$$\begin{aligned} \therefore v &= c + \frac{2a^2c}{b^2-a^2} = c \left[1 + \frac{2a^2}{b^2-a^2} \right] = c \left[\frac{b^2-a^2+2a^2}{b^2-a^2} \right] \\ &= c \left[\frac{b^2+a^2}{b^2-a^2} \right] \end{aligned}$$

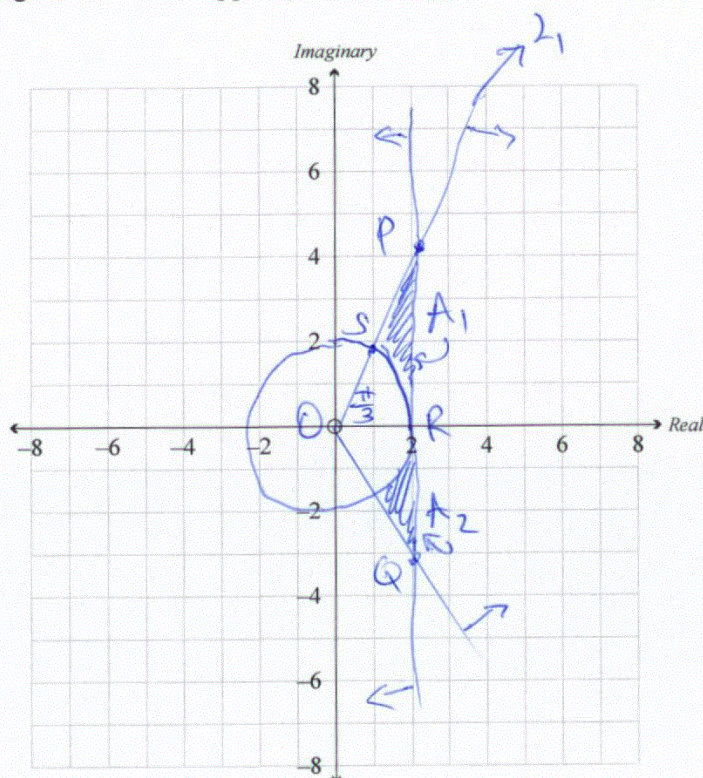
QUESTION FOUR

(a) Consider all the complex numbers z that satisfy all of the following three conditions:

- $-\frac{\pi}{3} \leq \arg(z) \leq \frac{\pi}{3}$
- $z + \bar{z} \leq 4$
- $|z| \geq 2$

Find the exact area of the region generated in an Argand diagram by the locus of points that represent z .

Use the Argand diagram below to support your working.



$$z + \bar{z} \leq 4 \Leftrightarrow \frac{z + \bar{z}}{2} \leq 2 \Leftrightarrow \operatorname{Re}(z) \leq 2$$

line L_1 is $y = \sqrt{3}x$, so point P is $(2, 2\sqrt{3})$.

$$\text{Area } \triangle OPR = \frac{1}{2} \cdot \text{base} \cdot \text{height} = \frac{1}{2} \cdot 2 \cdot 2\sqrt{3} = 2\sqrt{3}$$

$$\text{Area (sector } \widehat{OSR}) = \frac{1}{2} r^2 \theta = \frac{1}{2} (2)^2 \left(\frac{\pi}{3}\right) = \frac{4\pi}{3}$$

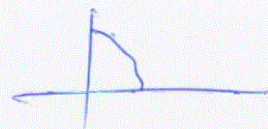
$$\text{Area } A_1 = 2\sqrt{3} - \frac{2\pi}{3}$$

By symmetry,

$$\begin{aligned} \therefore \text{Area of region} &= \text{total shaded} \\ &= 2A_1 = \underline{4\sqrt{3} - \frac{4\pi}{3}} \end{aligned}$$

(b) If $z = \cos\theta + i\sin\theta$ and $0 \leq \theta \leq \frac{\pi}{2}$, prove the following:

~~z = e^{i\theta}~~



(i) $\frac{1}{1-z\cos\theta} = 1 + i\cot\theta$

~~LHS = \frac{1}{1-z\cos\theta} = \frac{1}{1-(\cos\theta + i\sin\theta)\cos\theta}~~

$$\text{LHS} = \frac{1}{1-z\cos\theta} \times \frac{1-\bar{z}\cos\theta}{1-\bar{z}\cos\theta}$$

$$= \frac{1-\bar{z}\cos\theta}{1-\cos\theta(z+\bar{z}) + z\bar{z}\cos^2\theta}$$

$$= \frac{1-\bar{z}\cos\theta}{1-\cos\theta(\cos\theta + i\sin\theta + \cos\theta - i\sin\theta) + \cos^2\theta}$$

$$= \frac{1-\bar{z}\cos\theta}{1-2\cos^2\theta + \cos^2\theta} = \frac{1-\bar{z}\cos\theta}{1-\cos^2\theta}$$

$$= \frac{1-\bar{z}\cos\theta}{\sin^2\theta} = \frac{1-(\cos\theta - i\sin\theta)\cos\theta}{\sin^2\theta}$$

$$= \frac{1-\cos^2\theta + i\sin\theta\cos\theta}{\sin^2\theta}$$

$$= \frac{\sin^2\theta + i\sin\theta\cos\theta}{\sin^2\theta} = 1 + i \frac{\sin\theta\cos\theta}{\sin^2\theta}$$

$$= 1 + i\cot\theta = \text{RHS. } \times$$

(ii) $2\arg(z+1) = \arg(z)$

~~LHS = 2\arg(z+1) = \arg(z+1) + \arg(z+1)~~

We will prove instead that $2\arg(z+1) - \arg(z) = 0$.

$$2\arg(z+1) - \arg(z)$$

$$= \arg[(z+1)^2] - \arg(z)$$

$$= \arg\left[\frac{(z+1)^2}{z}\right] = \arg\left[\frac{z^2 + 2z + 1}{z}\right]$$

$$= \arg\left[z + 2 + \frac{1}{z}\right]$$

$$= \arg[z + \bar{z} + 2]$$

$$= \arg[2\cos\theta + 2]$$

$$= \arg[2\cos\theta + 2] = 0 \text{ because } 2\cos\theta + 2 \text{ is a positive real number.}$$

$$\arg(zw) = \arg z + \arg w$$

The use of this is justified by $0 \leq \theta \leq \frac{\pi}{2}$ here.

~~\times~~

(directrix)

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(focus)

(c) The point $P(2p, p)$ is some point on the line $y = \frac{1}{2}x$ where $p \geq 0$.

- (i) Consider the locus of points that are the same distance from P as they are from the line $y = -2x$.

Explain why this locus is a parabola AND clearly describe its key features.

Note: you do not need to find the equation of the parabola.

This is a parabola as the given locus can be classified as a ^{conic sections} locus of the type "locus of points S such that the ratio $\frac{PS}{d}$, where P is a fixed point and d is distance to fixed line, is equal to " e ", the eccentricity". It is known that when $e=1$, as it is in this case, the ~~locus~~ conic section locus will be a parabola. In this case, the directrix is $y = -2x$, the focus of the parabola is $P(2p, p)$, the vertex of the parabola will lie on $y = \frac{1}{2}x$ (as $y = \frac{1}{2}x$ and $y = -2x$ are $\perp$), and the axis of symmetry will be $y = \frac{1}{2}x$.

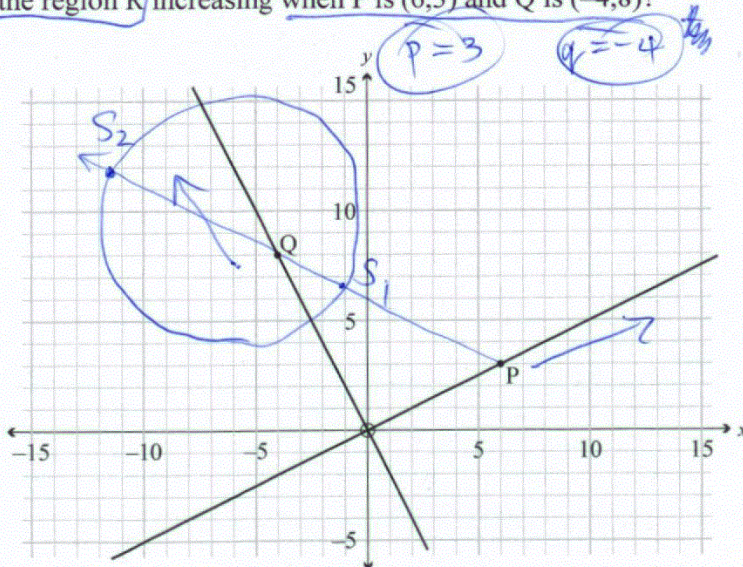
- (ii) Consider another point $Q(q, -2q)$ on the line $y = -2x$ where $q \leq 0$.

Let R be the region enclosed by the locus of points that are three times as far from P as they are from the point Q . Apollonius circle?

Now suppose the points P and Q are moving along their respective lines.

If p is increasing at a rate of 3 cm s^{-1} , and q is decreasing at a rate of 2 cm s^{-1} , at what rate is the area of the region R increasing when P is $(6, 3)$ and Q is $(-4, 8)$?

in fact, the vertex will be the midpoint of P and $(0, 0)$, i.e. $(p, \frac{p}{2})$.



The locus is a circle whose centre lies on PQ .
To prove this, ~~the~~ $S = (x, y)$ $\longrightarrow$

SP

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diameter $\div 2 = \text{radius}$

$$\sqrt{(x-2p)^2 + (y-p)^2} = 3 \sqrt{(x-q)^2 + (y+2q)^2}$$

$$(x-2p)^2 + (y-p)^2 = 9 [(x-q)^2 + (y+2q)^2]$$

It can be clearly seen that this is a circle, upon completing further algebra, something of the form $x^2 + y^2 + Bx + Cy + D = 0$ will be reached.

Now consider the diameter marked $S_1 S_2$ on the previous page. We have:

$$QS_1 = \frac{1}{4} PQ.$$

$$QS_2 = \frac{1}{3} (PS_2) = \frac{1}{3} (PQ + QS_2)$$

$$3QS_2 = PQ + QS_2 \Rightarrow QS_2 = \frac{1}{2} PQ.$$

Thus the radius ^{squared} of the circle is then:

$$r^2 = \left(\frac{QS_1 + QS_2}{2} \right)^2 = \left(\frac{\frac{1}{4} PQ + \frac{1}{2} PQ}{2} \right)^2 = \left(\frac{3}{8} PQ \right)^2 = \frac{9PQ^2}{64}.$$

Hence the area of the circle is $\pi r^2 = \pi \cdot \frac{9PQ^2}{64}$.

So we need to find the rate at which PQ^2 is changing.

$$PQ^2 = (2p-q)^2 + (p+2q)^2 = 5p^2 + 5q^2.$$

$$PQ = \sqrt{5p^2 + 5q^2} = \sqrt{5(3^2 + 4^2)} = 5\sqrt{5}.$$

$$\frac{d(PQ^2)}{dt} = (5pp' + 5qq') \times 2 = 10 \times (3 \times 3 + 4 \times (-2)) = 170$$

$$2PQ \frac{dPQ}{dt} = 170$$

$$\frac{dPQ}{dt} = \frac{85}{PQ} = \frac{85}{5\sqrt{5}} = \frac{17}{\sqrt{5}} = \frac{17\sqrt{5}}{5}$$

$$\frac{dA}{dt} = \frac{d(\pi r^2)}{dt} = \frac{9\pi}{64} \cdot \frac{d(PQ^2)}{dt} = \frac{9\pi \times 170}{64}$$

$$= \frac{765\pi}{32} \text{ cm}^2 \text{ s}^{-1} \text{ (increasing)}$$

Extra space if required.
Write the question number(s) if applicable.

QUESTION
NUMBER

2c ii Thus we can find the minimum simply by substituting in, say $x = 1012.5$.

$$(S_{2024}(x))_{\min} = |1012.5 - 1| + \dots + |1012.5 - 1012| \\ + |1012.5 - 1013| + \dots + |1012.5 - 2024| \\ = 2 [0.5 + 1.5 + \dots + 1011.5]$$

$$= 1 + 3 + 5 + \dots + 2023$$

$$= \overset{1012}{2023} + (0 + 2 + 4 + \dots + 2022)$$

$$= \overset{1012}{2023} + 2 \times (1 + 2 + 3 + \dots + 1011)$$

$$= \overset{1012}{2023} + 2 \times \left(\frac{1011 \times 1012}{2} \right)$$

$$= \cancel{2023} \cancel{1011} \cancel{1012}$$

$$= \underline{\underline{1024144}}$$

$$\swarrow \\ = 1012^2.$$

Since $\sum_{n=1}^{n=k} n = \frac{k(k+1)}{2}$,
see 2023 Top Scholar script
for proof.

Extra space if required.
Write the question number(s) if applicable.

QUESTION
NUMBER

3c ii ^{so} We have $u = \frac{2abc}{b^2 - a^2}$ and $v = c \left[\frac{b^2 + a^2}{b^2 - a^2} \right]$.

But recall from part (i) that

$$c^2 = \frac{2ab}{\sin 2\alpha} \Rightarrow \sin 2\alpha = \frac{2ab}{c^2}.$$

Now considering basic right-angled trig in $\triangle XYW$, we have $\sin(2\alpha) = \frac{u}{v}$, so thus we have:

$$\frac{2ab}{c^2} = \frac{\left(\frac{2abc}{b^2 - a^2} \right)}{\left(c \cdot \left[\frac{b^2 + a^2}{b^2 - a^2} \right] \right)}$$

$$\Rightarrow \frac{2ab}{c^2} = \frac{2ab}{a^2 + b^2}$$

$$\Rightarrow a^2 + b^2 = c^2, \text{ as required.} //$$