S

93202Q

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New Zealand Qualifications Authority

Scholarship 2023
Calculus

Time allowed: Three hours
Total score: 32

QUESTION BOOKLET

There are four questions in this booklet. Answer ALL FOUR questions.

Each question is equally weighted.

If in any question you make additions to a diagram and refer to those additions in your solution, the diagram must be replicated in your answer booklet as part of your solution.

Write your answers in Answer Booklet 93202A.

Pull out Formulae and Tables Booklet S–CALCF from the centre of this booklet.

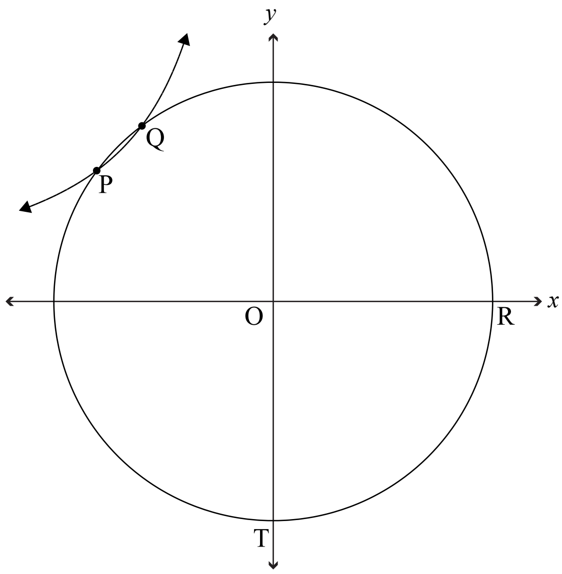
Check that this booklet has pages 2–5 in the correct order and that none of these pages is blank.

YOU MAY KEEP THIS BOOKLET AT THE END OF THE EXAMINATION.

QUESTION ONE

(a) The graph below shows a circle and a hyperbola.

The circle has the equation $x^2 + y^2 = r^2$, and the hyperbola is defined by $xy = k$ for $x < 0$.



The points P and Q are where the circle intersects with the hyperbola, and points R and T are where the circle intersects with the positive x-axis and negative y-axis respectively.

(i) If P has coordinates of $(-8,6)$, find the equation of the line QR.

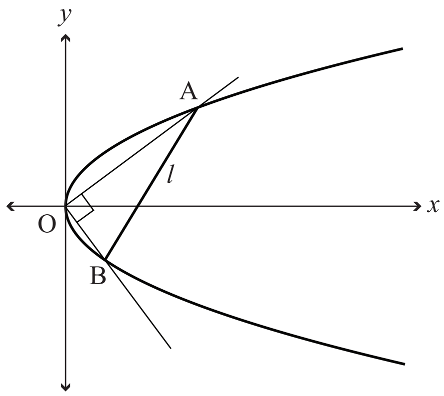
(ii) Prove that the line RT is parallel to the line PQ no matter where the hyperbola intersects with the circle.

(b) Solve the system:

$$4 \log_2(8x^3) + \log_5(y^6) = 17$$

$$\log_2(x^5) + \log_5(y^2) = 3$$

(c) Line l intersects the parabola $y^2 = kx$ at points A and B, such that OA is perpendicular to OB, as shown.



Show that line l crosses the x-axis at $(k,0)$.

QUESTION TWO

(a) Given that $\frac{3\pi}{2} < x < 2\pi$ and $\cot(x) = -\frac{5}{12}$, find the exact value for $\sin\left(\frac{x}{2}\right)$.

(b) Given $z = -\frac{1}{2} + \frac{\sqrt{3}}{2}\mathrm{i}$, find $(\overline{z})^4$ in rectangular form.

(c) Euler discovered an exponential equivalent polar form for complex numbers. He found that any complex number $z = r\operatorname{cis}\theta$ could be written in the form $z = r\mathrm{e}^{\mathrm{i}\theta}$.

From this, it can be shown that $\mathrm{e}^{\mathrm{i}\theta} = \cos\theta + \mathrm{i}\sin\theta$.

Use this identity, or otherwise to:

(i) Determine the exact **real** value for $\left(\mathrm{i}^{\mathrm{i}}\right)^2$.

(ii) Write the complex number $\ln\left(-25\,\mathrm{e}^{\mathrm{i}^{\mathrm{i}}}\right)$ in exact rectangular form.

(d) The general equation of a circle is $A(x^2 + y^2) + Bx + Cy + D = 0$.

If $z = x + \mathrm{i}y$ is a complex number, show that any circle can be written in the form:

$$\alpha z\overline{z} + \beta z + \overline{\beta}\overline{z} + \gamma = 0$$

where α and γ are real constants, but β may be non-real.

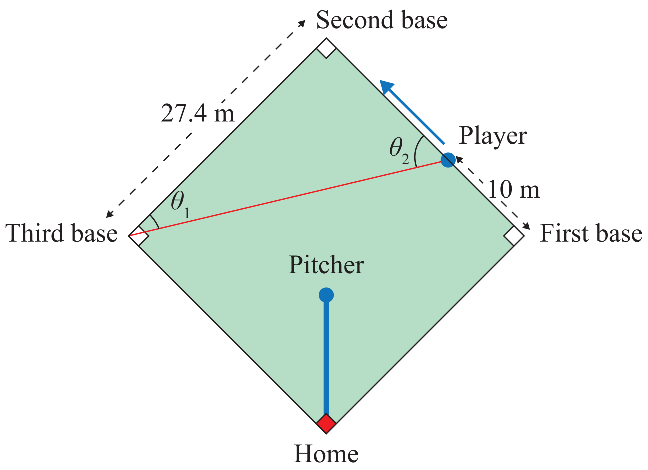
QUESTION THREE

(a) Consider the curve defined parametrically by the equations:

$$x = 2t - 3t^3$$
$$y = te^t$$

Find the coordinates of any stationary points on the curve, and determine their nature.

(b) A baseball diamond is a square with side length 27.4 m, as shown.



A player runs from first base to second base at a rate of 5 m s^{-1}.

(i) At what rate is the player's distance from third base changing when the player is 10 metres from first base?

(ii) At what rate are angles θ_1 and θ_2 changing as the player slides into second base at the rate of 5 m s^{-1}.

(c) A gate manufacturer uses steel to produce five gates per day.

Delivery for any amount of steel ordered is a fixed price of $5000. If the manufacturer orders a large quantity of steel each time, this will reduce delivery costs per unit, but will increase storage costs.

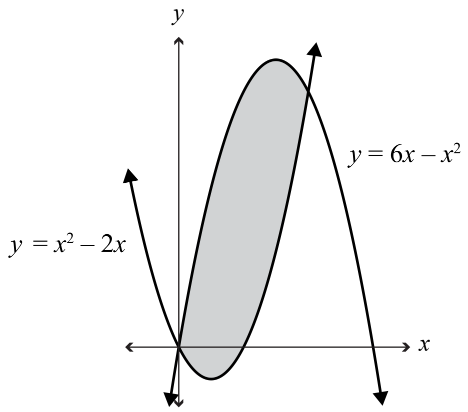
It costs $10 per day to store one unit of steel, where a unit is the amount of steel needed to produce one gate.

How often should the manufacturer order steel, and how much should they order each time, to minimise the average daily cost of delivery and storage for each production cycle?

You may assume that no completed gates are stored.

QUESTION FOUR

(a) Find the area of the region enclosed by the parabolas $y = 6x - x^2$ and $y = x^2 - 2x$.



$y =$

(b) Evaluate the definite integral $\displaystyle\int_1^4 \frac{\mathrm{d}x}{(x-2)^{\frac{2}{3}}}$.

Hint: consider using limits, i.e. $\displaystyle\int_a^b f(x)\,\mathrm{d}x = \lim_{k \to a^+} \int_k^b f(x)\,\mathrm{d}x$.

(c) When a curve is defined parametrically by the equations $x = f(t)$ and $y = g(t)$ on an interval $t_1 \le t \le t_2$, we can find the length of the curve using the formula:

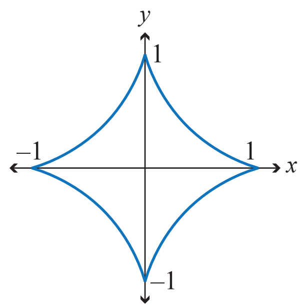
$$L = \int_{t_1}^{t_2} \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t$$

An astroid is the curve produced by a point on a circle as it rolls around the inside of a fixed circle with four times the radius. The astroid shown alongside is defined by the parametric equations:

$x = \cos^3 t$

$y = \sin^3 t$

$0 \le t \le 2\pi$



Find the perimeter of the astroid.

(d) A function $f(x)$, where $x \ge 0$, is defined implicitly by the following formula:

$$f(x) = \sqrt{\int_0^x \left(f(t)^2 + f'(t)^2\right)\mathrm{d}t + 2023}$$

Find an explicit expression for $f(x)$ in its simplest form.

Hint: If $f(x)$ is continuous on an interval $a \le x \le b$, then for all x:

$$F(x) = \int_a^x f(t)\,\mathrm{d}t \text{ and } F'(x) = f(x).$$

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