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Draw a cross through the box (☒) if you have NOT written in this booklet

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SCHOLARSHIP EXEMPLAR

Mana Tohu Mātauranga o Aotearoa
New Zealand Qualifications Authority

Scholarship 2023
Calculus

Time allowed: Three hours
Total score: 32

ANSWER BOOKLET

Check that the National Student Number (NSN) on your admission slip is the same as the number at the top of this page.

Write your answers in this booklet.

Make sure that you have Formulae Booklet S–CALCF.

Show ALL working. Start your answer to each question on a new page. Carefully number each question.

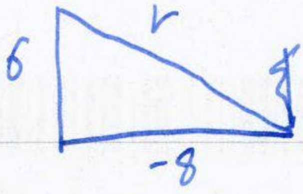
Answers developed using a CAS calculator require **ALL commands to be shown**. Correct answers only will not be sufficient.

If in any question you make additions to a diagram and refer to those additions in your solution, the diagram must be replicated in this booklet as part of your solution.

Check that this booklet has pages 2–27 in the correct order and that none of these pages is blank.

Do not write in any cross-hatched area. This area may be cut off when the booklet is marked.

YOU MUST HAND THIS BOOKLET TO THE SUPERVISOR AT THE END OF THE EXAMINATION.



1a) i) $r = \sqrt{6^2 + (-8)^2}$

$= \sqrt{100}$

$= 10$

$\therefore$ equation of circle is $x^2 + y^2 = 100$

equation of hyperbola : $y = \frac{k}{x}$

subbing in known point P : $6 = \frac{k}{-8}$

$k = -48$

$\therefore$ equation of hyperbola is $y = \frac{-48}{x}$

to find other intersection point Q

$x^2 + y^2 - 100 = xy + 48$

$x^2 + xy + y^2 = 148$

subbing in hyper eq. $x^2 + x \cdot \frac{-48}{x} + \left(\frac{-48}{x}\right)^2 = 148$

$x^2 - 48 + \frac{2304}{x^2} = 148$

$x^2 + \frac{2304}{x^2} = 196$

$\frac{1}{x^2}(x^4 + 2304) = 196$

$x^4 + 2304 = 196x^2$

$x^4 - 196x^2 + 2304 = 0$

let $v = x^2$ $\quad v^2 - 196v + 2304 = 0$

$v =$

Q : $x = -6 \quad y = 8$

R : $x = 10 \quad y = 0$

$\frac{\Delta y}{\Delta x} = \frac{0-8}{10 - -6} = \frac{-8}{16} = \frac{-1}{2}$

$y - y_1 = m(x - x_1)$

$y - 8 = -\frac{1}{2}(x - -6)$

$y - 8 = -\frac{1}{2}x - 3$

Equation of line QR

$y = -\frac{1}{2}x + 5$

1c) $x = \frac{y^2}{k}$ $m_{OA} = -(m_{OB})^{-1}$

$y = \sqrt{kx}$ $m_{OA} - m_{OB} = m_l$

$m_{OA} = \frac{y}{x} = \frac{y}{\frac{y^2}{k}} = \frac{k}{y}$

$m_{OB} = -\left(\frac{k}{y}\right)^{-1} = \frac{-y}{k}$

$\frac{k}{y} - -\frac{y}{k} = m_l$

$ml = \frac{k^2+y^2}{ky}$

$y - \sqrt{kx} = \frac{k^2+y^2}{ky}\left(x - \frac{y^2}{k}\right)$

$y - \sqrt{kx} = \frac{k^2+y^2}{ky}x - \frac{y^2(k^2+y^2)}{k^2y}$

2a) $\cancel{\frac{1}{\tan\theta} = -\frac{5}{12}}$ $\frac{1}{\tan x} = \frac{-5}{12}$

$\frac{\cos x}{\sin x} = \frac{-5}{12}$ $\tan x = -\frac{12}{5}$

$\cancel{x = }$

$\frac{\sin x}{\cos x} = \frac{-12}{5}$

$\sin x = \frac{-12\cos x}{5}$

2b) $\bar{z} = -\frac{1}{2} - \frac{\sqrt{3}}{2} i$

$(\bar{z})^4 = \left(-\frac{1}{2} - \frac{\sqrt{3}}{2} i\right)^4$

$= \left(-\frac{1}{2} - \frac{\sqrt{3}}{2} i\right)^2 \left(-\frac{1}{2} - \frac{\sqrt{3}}{2} i\right)^2$

$= \left(\frac{1}{4} + \frac{2\cdot\sqrt{3}}{4} i - \frac{3}{4}\right)^2$

$= \left(-\frac{1}{2} + \frac{\sqrt{3}}{2} i\right)^2$

$\therefore = -\frac{1}{2} - \frac{\sqrt{3}}{2} i$

2c)i $z = i^i$

$\arg(z) = \frac{\pi}{2}$

$r = \sqrt{0^2 +}$

~~2d~~

3a) $\frac{dx}{dt} = 2 - 9t^2 \quad \frac{dy}{dt} = te^t + 1e^t$

$\frac{dy}{dx} = \frac{(t+1)e^t}{2-9t^2} = 0$

$(t+1)e^t = 0$

$e^t \neq 0$ as $\ln 0$ is undefined

$\therefore t = -1$

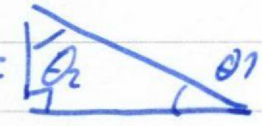
Mi

$$\frac{d^2y}{dx^2} = \frac{((t+1)\cdot(e^t) + t\cdot e^t)(2-9t^2) - (te^t e^t)(-18t)}{(2-9t^2)^2} \cdot \frac{1}{(2-9t^2)}$$

$t = -1 = 0.040089713 \quad \frac{d^2y}{dx^2} > 0 \therefore t = -1$ is min-pt

coords @ $x = 2 \cdot -1 - 3(-1)^3$
$= -2 + 3$
$= 1$
$y = -1e^{-1}$
$= -0.3678$
$(1, -0.3678)$ minimum point

3b) i)



$D = \sqrt{27.4^2 + x^2}$

$\frac{dD}{dx} = \frac{\cancel{2}x}{\cancel{2}\sqrt{27.4^2+x^2}} = \frac{x}{\sqrt{27.4^2+x^2}}$

$\frac{dx}{dt} = -5ms$ $\quad x = 27.4 - 10 = 17.4m$

$\frac{dD}{dt} = \frac{-5x}{\sqrt{27.4^2+x^2}} = \frac{-5 \cdot 17.4}{\sqrt{27.4^2+17.4^2}} = \frac{\cancel{\sqrt{5213.86}}\ 87}{32.45717}$

$= \cancel{46.6387 ms}$
$= \cancel{46.6 ms^{-1}}$

$\frac{dD}{dt} = -2.69 ms^{-1}$

ii) $\theta_1 = \tan^{-1}\left(\frac{x}{27.4}\right)$ $\quad \theta_1 = 0$ $\quad \frac{dx}{dt} = -5ms^{-1}$
$x=0$

$\tan\theta_1 = \frac{x}{27.4}$

$\frac{dx}{d\theta_1} = 27.4\sec^2\theta_1$

$\frac{d\theta_1}{dx} = \frac{1}{27.4\sec^2\theta_1}$

$\frac{d\theta_1}{dt} = \frac{-5}{27.4\sec^2\theta_1} = \frac{-5}{27.4\sec^2 0}$
$= \frac{-5}{27.4} = -0.18248$ rads^{-1}

$\theta_2 = \tan^{-1}\frac{27.4}{x}$ $\quad \lim_{x \to 0} \theta_2 = \frac{\pi}{2}$

$\tan\theta_2 = \frac{27.4}{x}$

$\frac{27.4}{\tan\theta_2} = x \qquad \frac{dx}{d\theta} = \frac{-27.4\sec^2\theta_2}{\tan^2\theta_2}$

$$\frac{d\theta}{dx} = \frac{-\tan^2\theta_2}{27.4\sec^2\theta_2}$$

$\left\{\frac{d\theta}{dx}\cdot\frac{dx}{dt}\right\}: \quad \frac{d\theta}{dt} = \frac{-\tan^2\theta_2}{27.4\sec^2\theta_2}\cdot\frac{-5}{1}$ $\quad \frac{\frac{\sin^2}{\cos^2}}{\frac{1}{\cos^2}}$

$$\frac{d\theta}{dt} = \frac{5\sin^2\theta_2}{27.4}$$

$$= \frac{5\cdot 1}{27.4}$$

Max $\quad = 0.18248 \text{ rad s}^{-1}$

4a) finding the iscts

$$x^2 - 2x - 6x + x^2 = 0$$

$$2x^2 - 8x = 0$$
$$2x(x-4) = 0$$
$$x = 0, x = 4$$

from $x=0$ to $x=2$

$A_{0-2} = \int_0^2 6x - x^2\,dx - \int_0^2 x^2 - 2x\,dx$

$= \left[3x^2 - \frac{x^3}{3}\right]_0^2 - \left[\frac{x^3}{3} - x^2\right]_0^2$

$= \frac{28}{3} - -\frac{4}{3} = \frac{32}{3}$ units^2

Gom 2-4

$A_{2-4} = \int_2^4 6x - x^2 - x^2 + 2x \, dx$

$= \int_2^4 8x - 2x^2 \, dx$

$= \left[4x^2 - \frac{2x^3}{3}\right]_2^4$

$= \frac{32}{3}$

$A_{tot} = \frac{32}{3} + \frac{32}{3} = \frac{64}{3}$ units^2

4b) $\int_1^4 (x-2)^{-2/3} \, dx$

$\left[3(x-2)^{1/3}\right]_1^4$ A

$= 3(4-2)^{1/3} - 3(1-2)^{1/3}$

$= 3(2)^{1/3} - 3(\sqrt[3]{-1})$

$= 3 \cdot \sqrt[3]{2} - 3 \cdot -1$

$= 3 \cdot \sqrt[3]{2} + 3.$ RAW

≈ 6.77976315

$-\frac{3}{4}(\cos t)^{x2}$

4C) $\frac{dx}{dt} = -3 \sin t \cos^2 t$

$\frac{dy}{dt} = 3 \cos t \sin^2 t$

using $x=-1$, $x=0$

bounds: $t = \pi$, $t = \frac{\pi}{2}$

Symmetry ∴ 4 identical sectors

$L = 4 \int_{\frac{\pi}{2}}^{\pi} \sqrt{(-3\sin t \cos^2 t)^2 + (3 \cos t \sin^2 t)^2}\, dt$

$L = 4 \int_{\frac{\pi}{2}}^{\pi} \sqrt{9 \sin^2 t \cos^4 t + 9 \cos^2 t \sin^4 t}\, dt$

$L = 4 \cdot 3 \int_{\frac{\pi}{2}}^{\pi} \sqrt{(\sin^2 t \cos^2 t)(\cos^2 t + \sin^2 t)}\, dt$

$L = 12 \int_{\frac{\pi}{2}}^{\pi} \sqrt{(\sin t \cos t)^2 \,(1)}\, dt$

$L = 12 \int_{\frac{\pi}{2}}^{\pi} \sin t \cos t \, dt$

$L = 6 \int_{\frac{\pi}{2}}^{\pi} 2 \sin t \cos t \, d\theta$

$L = 6 \int_{\frac{\pi}{2}}^{\pi} \sin 2t + \sin 0 \, dt$

$L = 6 \int_{\frac{\pi}{2}}^{\pi} \sin 2t \, dt$

$L = 6 \left[-\frac{\cos 2t}{2} \right]_{\frac{\pi}{2}}^{\pi}$

$L = 6 \left[-\frac{\cos 2\pi}{2} - -\frac{\cos \pi}{2} \right]$

$L = 6 \left(-\frac{1}{2} - \frac{1}{2} \right)$

$L = 6 \cdot -1 = -6$ (value is negative as bounds in negative x-region were used. Actual perimeter is $+6$ units.)

Perimeter = 6 units

Scholarship

Subject: Calculus

Standard: 93202

Total score: 24

Q	Score	Marker commentary
1	05	The candidate showed ability in transforming a quartic equation into a quadratic equation and then using it to find the point ‘Q’ in Q1ai. They displayed strong algebra skills in ‘prove’ the two lines are parallel in part Q1aii, although it could have been done by symmetry more easily. They also showed proficiency in dealing with logarithms in solving simultaneous equations in Q1b.
2	08	The candidate worked with fluency in manipulating trig identities and the compound angle formulae in Q2a. They also demonstrated an excellent grasp of complex numbers by adeptly expressing them in exponential form and successfully navigated and solved problems with minimal descriptions in Q2c.
3	06	The candidate was able to find the second derivative of a parametric function accurately in Q3a. They successfully constructed a mathematical model and used it to find the related rates of change problem in 3bi. They also applied implicit differentiation in Q3bii correctly, and could have solved the problem if they had substituted the correct angle in.
4	05	The candidate showed ability in finding area between curves accurately using integration in Q4a. They managed to integrate correctly in Q4c and could have been successful if they used the new integral limits after the transformation.