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Draw a cross through the box (☒) if you have NOT written in this booklet

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OUTSTANDING SCHOLARSHIP EXEMPLAR

Mana Tohu Mātauranga o Aotearoa
New Zealand Qualifications Authority

Scholarship 2023
Calculus

Time allowed: Three hours
Total score: 32

ANSWER BOOKLET

Check that the National Student Number (NSN) on your admission slip is the same as the number at the top of this page.

Write your answers in this booklet.

Make sure that you have Formulae Booklet S–CALCF.

Show ALL working. Start your answer to each question on a new page. Carefully number each question.

Answers developed using a CAS calculator require **ALL commands to be shown**. Correct answers only will not be sufficient.

If in any question you make additions to a diagram and refer to those additions in your solution, the diagram must be replicated in this booklet as part of your solution.

Check that this booklet has pages 2–27 in the correct order and that none of these pages is blank.

Do not write in any cross-hatched area ( ). This area may be cut off when the booklet is marked.

YOU MUST HAND THIS BOOKLET TO THE SUPERVISOR AT THE END OF THE EXAMINATION.

1.a.i.

$(-8)^2 + 6^2 = r^2$

$r^2 = 100$

$r = 10$ WLOG

$(-8) \cdot 6 = k$

$k = -48$

$y = \frac{-48}{x}$

$x^2 + \frac{2304}{x^2} = 100$

$x^4 - 100x^2 + 2304 = 0$

$x^2 = \frac{100 \pm \sqrt{784}}{2}$

$x^2 = 50 \pm 14$

$x = \pm 6$ or $x = \pm 8$

Clearly $x < 0$, $x = -8$ is point P so point Q has

$x = -6 \Rightarrow y = 8$

$Q = (-6, 8)$

$R = (10, 0)$ — Circle with radius 10

Gradient $m = \frac{0-8}{10-(-6)} = -\frac{1}{2}$

$y = -\frac{1}{2}x + c$

$0 = -\frac{1}{2} \cdot 10 + c$

$c = 5$

$y = -\frac{1}{2}x + 5$.

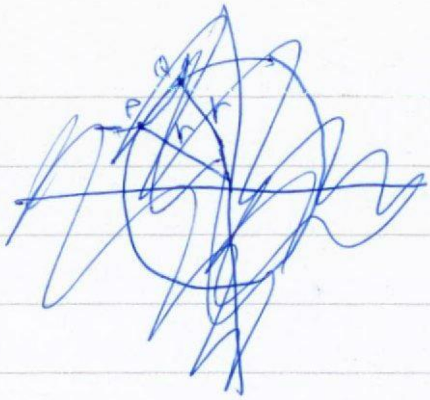
1a.ii. $y=\frac{k}{x}$

$x^2+\frac{k^2}{x^2}=r^2$

$x^4-r^2x^2+k^2=0$

$x^2=\frac{r^2\pm\sqrt{r^4-4k^2}}{2}$

$x=\pm\sqrt{\frac{r^2\pm\sqrt{r^4-4k^2}}{2}}$



The two points P and Q will have x values $-\sqrt{\frac{r^2\pm\sqrt{r^4-4k^2}}{2}}$

as both x values are negative

If we consider the sum of squares,

$x_P^2+x_Q^2=\frac{r^2+\sqrt{r^4-4k^2}}{2}+\frac{r^2-\sqrt{r^4-4k^2}}{2}=r^2$

Since both points lie on the circle,

$x_P^2+y_P^2=r^2$ x values are negative and

$x_Q^2=y_P^2$ y values are positive so

also $x_Q^2+y_Q^2=r^2$ $x_Q=-y_P$

$x_P^2=y_Q^2$ $x_P=-y_Q$

The gradient of line PQ, $m_1=\frac{y_Q-y_P}{x_Q-x_P}$

$m_1=\frac{y_Q+x_Q}{x_Q+y_Q}=1$

$R=(r,0)$ For line RT

$T=(0,-r)$ $\Rightarrow m_2=\frac{-r-0}{0-r}=1 \Rightarrow m_1=m_2$ so RT is parallel to PQ.

1.6. $\cancel{4\log_2(8) + 4\log_2(x^3) + 2\log_5(y^3) = 17}$

$4\log_2(8) + 4\log_2(x^3) + 6\log_5(y) = 17$ } First

$12\log_2(x) + 6\log_5(y) = 5$

$5\log_2(x) + 2\log_5(y) = 3$ } Second

$15\log_2(x) + 6\log_5(y) = 9$

$3\log_2(x) = 4$ Taking Difference

$\log_2(x) = \frac{4}{3}$

$x = \sqrt[3]{16}$

$5\log_2(\sqrt[3]{16}) + 2\log_5(y) = 3$

$2\log_5(y) = -\frac{11}{3}$

$y = 5^{-\frac{11}{6}}$ //

2.a. ~~$\operatorname{cosec}^2(x) = \cot^2(x)+1 = \frac{169}{144}$~~

$\tan(x) = -\frac{12}{5}$

$\sec^2(x) = \tan^2(x)+1 = \frac{169}{25}$

$\cos^2(x) = \frac{25}{169}$

$\cos(x) = \pm \frac{5}{13}$

Given $\frac{3\pi}{2} < x < 2\pi \Rightarrow \cos(x) \geq 0$

$\cos(x) = \frac{5}{13}$.

$2\sin^2(\frac{x}{2}) = 1-\cos(x) = \frac{8}{13}$.

$\sin^2(\frac{x}{2}) = \frac{4}{13}$

$\sin(\frac{x}{2}) = \pm \frac{2}{\sqrt{13}}$

Given $\frac{3\pi}{4} < \frac{x}{2} < \pi \Rightarrow \sin(\frac{x}{2}) \geq 0$

$\sin(\frac{x}{2}) = \frac{2}{\sqrt{13}}$. //

2.b. $z = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$

$-z = \sqrt{(-\frac{1}{2})^2+(-\frac{\sqrt{3}}{2})^2}\left(\cos\left(\tan^{-1}\left(\frac{-\frac{\sqrt{3}}{2}}{-\frac{1}{2}}\right)\right)+i\sin\left(\tan^{-1}\left(\frac{-\frac{\sqrt{3}}{2}}{-\frac{1}{2}}\right)\right)\right)$

$-z = \cos(\frac{\pi}{3})+i\sin(\frac{\pi}{3})$. as $Re(z), Im(z) < 0$

$z = \cos(-\frac{2}{3}\pi)+i\sin(-\frac{2}{3}\pi)$

$(z)^4 = \cos(-\frac{8}{3}\pi)+i\sin(-\frac{8}{3}\pi)$

$(z)^4 = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$. //

2.c.i. $i^i = e^{i\ln(i)}$

$i = \cos(\frac{\pi}{2})+i\sin(\frac{\pi}{2}) = e^{i\frac{\pi}{2}}$.

$\ln(i) = \frac{\pi}{2}i$

$i^i = e^{\frac{\pi}{2}i \cdot i} = e^{-\frac{\pi}{2}}$.

$(i^i)^2 = e^{-\pi}$. //

ii. $\ln(-25e^{i^i}) = \ln(-25e^{e^{-\frac{\pi}{2}}})$

$-1 = e^{\pi i}$

$\ln(-25e^{i^i}) = \ln(25e^{e^{-\frac{\pi}{2}}+\pi i})$

$= \ln(25)+e^{-\frac{\pi}{2}}+\pi i$. //

3.a. $\frac{dx}{dt} = 2 - 9t^2$

$\frac{dy}{dt} = e^t + te^t$

$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{te^t + e^t}{2 - 9t^2} = 0$ Stationary Points

$te^t + e^t = 0$

$(t+1)e^t = 0$

$t = -1$

$x = 2(-1) - 3(-1)^3 = 1$

$y = -1 \cdot e^{-1} = -\frac{1}{e}$

$\frac{d^2y}{dx^2} = \frac{d\left(\frac{dy}{dx}\right)}{dx} = \frac{\left(\frac{d\left(\frac{dy}{dx}\right)}{dt}\right)}{\left(\frac{dx}{dt}\right)}$

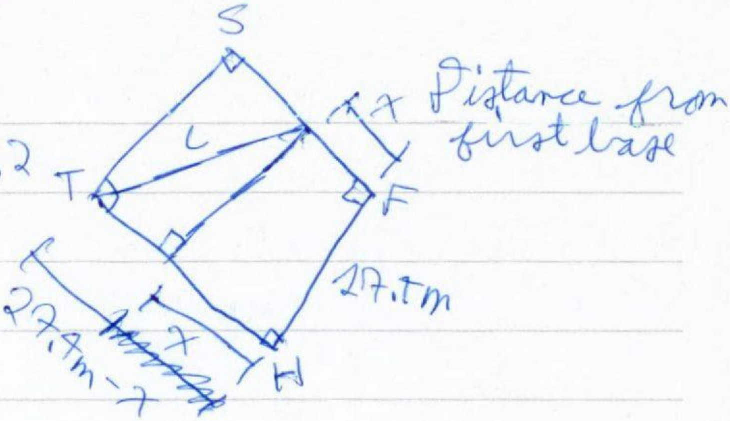
$\frac{d\left(\frac{dy}{dx}\right)}{dt} = \frac{te^t + 2e^t}{2 - 9t^2} + \frac{te^t + e^t}{(2 - 9t^2)^2} \cdot 18t$

$\frac{d^2y}{dx^2} = \frac{\frac{te^t + 2e^t}{2 - 9t^2} + 18t \cdot \frac{te^t + e^t}{(2 - 9t^2)^2}}{2 - 9t^2}$

at $t = -1$

$\frac{d^2y}{dx^2} = 4.508 \cdot 10^{-3} > 0$

Stationary Point $(1, -\frac{1}{e})$; Minimum



3.6.i. $l^2=(27.4-x)^2+27.4^2$

$l^2=x^2-54.8x+1501.52$

$2l\frac{dl}{dx}=2x-54.8$

$\frac{dl}{dx}=\frac{\cancel{2}x-27.4}{l}$

at $x=10m$

$l=\sqrt{10^2-54.8\cdot 10+1501.52}$

$l=\sqrt{1053.52}$

$\frac{dl}{dx}=\frac{-17.4}{\sqrt{1053.52}}$

$\frac{dx}{dt}=5$

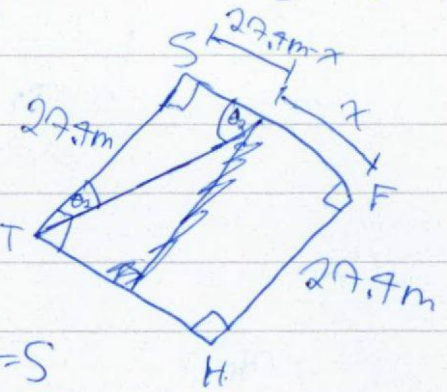
$\frac{dl}{dt}=\frac{dl}{dx}\cdot\frac{dx}{dt}=\frac{-87}{\sqrt{1053.52}}ms^{-1}=-2.68ms^{-2}$ (3sf)

ii. $\theta_2=\frac{\pi}{2}-\theta_1$

$\tan(\theta_1)=\cancel{\tan}\left(\frac{27.4-x}{27.4}\right)$

$\sec^2(\theta_1)\frac{d\theta_1}{dt}=-\frac{1}{27.4}\frac{dx}{dt}$

$=-\frac{25}{137}$ as $\frac{dx}{dt}=5$



$\frac{d\theta_1}{dt}=-\frac{25}{137}\cos^2(\theta_1)$

at second base, $\theta_1=0$

$\frac{d\theta_1}{dt}=-\frac{25}{137}$

$\frac{d\theta_2}{dt}=-\frac{d\theta_1}{dt}$

$\frac{d\theta_2}{dt}=\frac{25}{137}$

B, 4. $C_n = 5000 + 10 \cdot U_n$ where C_n is cost for one day, U_n is amount of units on that day

$U_n = U_{n-1} + B(n) - S$

Where $B(n)$ is the amount of units bought for day n.

~~$\frac{dU}{dn} = B(n) - S$~~ Let U_0 be the units on ~~first~~ zeroth day

~~$U_n = U_0 + n(B(n) - S)$~~ $U_n = U_0 + n(B - S)$

~~$U_n = \frac{dU}{dn} = B(n) - S$~~ $C_n = 5000 + 10(U_0 + n(B - S))$

~~$U_n = \int_0^n (B(t) - S)\,dt$~~

$C_n = \frac{10U_n + 5000}{U_n} = 10 + \frac{5000}{U_n}$

$U_n = U_{n-1} - S + B$ constant

$U_n = U_0 + n(B - S)$

$C_n = 10 + \frac{5000}{U_0 + n(B - S)}$

4.a. $6x - x^2 = x^2 - 2x$

$2x^2 - 8x = 0$

$x = 0$ or $x = 4$

~~By graphing~~ $6x - x^2 > x^2 - 2x$ across $x = [0,4]$

$A = \int_0^4 (6x - x^2 - (x^2 - 2x))\,dx$

$A = \int_0^4 (-2x^2 + 8x)\,dx$

$A = [-\frac{2}{3}x^3 + 4x^2]_0^4$

$A = \frac{64}{3}$

4.b. $\int_1^4 \frac{dx}{(x-2)^{\frac{2}{3}}} = \lim_{k\to 2^-}\left(\int_1^k (x-2)^{-\frac{2}{3}}\,dx\right) + \lim_{k\to 2^+}\left(\int_k^4 (x-2)^{-\frac{2}{3}}\,dx\right)$

$= \lim_{k\to 2^-}\left([3(x-2)^{\frac{1}{3}}]_1^k\right) + \lim_{k\to 2^+}\left([3(x-2)^{\frac{1}{3}}]_k^4\right)$

$= \lim_{k\to 2^-}(3\sqrt[3]{k-2} + 3) + \lim_{k\to 2^+}(3\sqrt[3]{2} - 3\sqrt[3]{k-2})$

$= 3 + 3\sqrt[3]{2}$

4.c. $\frac{dx}{dt} = -3\cos^2(t)\sin(t)$

$\frac{dy}{dt} = 3\sin^2(t)\cos(t)$

— By symmetry

$L = 4\int_0^{\frac{\pi}{2}} \sqrt{9\cos^4(t)\sin^2(t) + 9\sin^4(t)\cos^2(t)}\,dt$

$L = 4\int_0^{\frac{\pi}{2}} 3\sqrt{\cos^2(t)\sin^2(t)(\cos^2(t) + \sin^2(t))}\,dt$

$L = 4\int_0^{\frac{\pi}{2}} 3\cos(t)\sin(t)\,dt$

→ Next Page

4.c. Continued

$L = \cancel{\int \cos(t)\sin(t)\,dt}\ 12\int_0^{\frac{\pi}{2}} \cos(t)\sin(t)\,dt$

let $u = \sin(t)$

$du = \cos(t)\,dt$

$L = \cancel{8}\ 12\int_0^1 u\,du$

$L = 12\left[\frac{u^2}{2}\right]_0^2$

$L = 6$

4.d. $(f(x))^2 = \int_0^x ((f(t))^2 + (f'(t))^2)\,dt + 2023$

Differentiating with respects to x

$f'(x) \cdot 2f(x) \cancel{\frac{df}{dx}} = (f(x))^2 + (f'(x))^2$

$(f'(x))^2 - 2f(x)f'(x) + (f(x))^2 = 0$

$(f'(x) - f(x))^2 = 0$

$f'(x) = f(x)$

$\cancel{f(x)}\ \frac{f'(x)}{f(x)} = 1$

$\int \frac{f'(x)}{f(x)}\,dx = \int dx$

$\ln|f(x)| = x + c$

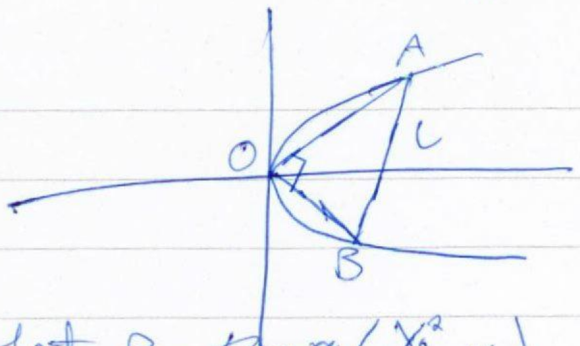
$f(x) = f(0)e^x$

$f(0) = \sqrt{\int_0^0 ((f(t))^2 + (f'(t))^2)\,dt + 2023}$

$f(0) = \sqrt{2023}$

$f(x) = \sqrt{2023}\,e^x$

1.c.



let $B = (\frac{y_0^2}{k}, y_0)$

Line BO has gradient $m_1 = \frac{y_0}{\left(\frac{y_0^2}{k}\right)} = \frac{k}{y_0}$

Line OA is perpendicular so has gradient

$m_2 = -\frac{1}{m_1} = -\frac{y_0}{k}$

Line OA crosses origin so equation is $y = -\frac{y_0}{k}x$

This intersects parabola when:

$y = -\frac{y_0}{k} \cdot \frac{y^2}{k}$

$y = -\frac{k^2}{y_0}$

$A = (\frac{k^3}{y_0^2}, -\frac{k^2}{y_0})$

Gradient of line l is $m_3 = \frac{-\frac{k^2}{y_0} - y_0}{\frac{k^3}{y_0^2} - \frac{y_0^2}{k}}$

$m_3 = \frac{-k^3y_0 - ky_0^3}{k^4 - y_0^4}$

$m_3 = \frac{ky_0(k^2+y_0^2)}{(y_0^2-k^2)(y_0^2+k^2)} = \frac{ky_0}{(y_0-k)(y_0+k)}$

Line l intersects point $(\frac{y_0^2}{k}, y_0)$

$y = \frac{ky_0}{(y_0-k)(y_0+k)}x + c$

$y_0 = \frac{y_0^3}{(y_0-k)(y_0+k)} + c$

$c = y_0\left(\frac{y_0^2-k^2}{y_0^2-k^2} - \frac{y_0^2}{y_0^2-k^2}\right) = \frac{y_0k^2}{k^2-y_0^2}$

$y = \frac{ky_0}{y_0^2-k^2}x + \frac{y_0k^2}{k^2-y_0^2}$

At $y = 0$

$x \cdot \frac{ky_0}{y_0^2-k^2} = -\frac{y_0k^2}{k^2-y_0^2}$

$x = k$

l crosses x axis at $(k, 0)$

2.d. $z\bar{z} = (x+iy)(x-iy) = x^2+y^2$

$\cancel{\alpha(x^2+y^2)}\ \alpha z\bar{z} + \beta z + \bar{\beta}\bar{z} + \gamma = 0$

$\alpha(x^2+y^2) + \beta z + \bar{\beta}\bar{z} + \gamma = 0$

Let $\beta = a+bi$

$\beta z = (a+bi)(x+iy) = ax - by + (bx+ay)i$

$\bar{\beta}\bar{z} = (a-bi)(x-iy) = ax - by - (bx+ay)i$

$\alpha(x^2+y^2) + 2ax - 2by + \gamma = 0$

Equivalent to circle where $A = \alpha$,

$B = 2a = 2\mathrm{Re}(\beta)$, $C = -2b = -2\mathrm{Im}(\beta)$, $D = \gamma$. //

3.c. Let D be the number of days between each order and thus the length of production cycle. At the end of production cycle no units should be left as these would not be sold and would thus be a waste of storage. At the first day they should order 5D units to have enough to sell for entire production cycle, but no extra.

The cost on the first day $C_1 = 5000 + 50D$

For every subsequent day 5 units are removed at day n amount of units is $5D - 5(n-1)$

$\therefore C_n = 10(5D - 5(n-1)) = 50(D-n+1)$

The total cost C for the production cycle $C = \sum_{k=1}^{D} C_k$

$C = C_1 + \sum_{k=2}^{D} 50(D-k+1)$

$C = 5000 + 50D + \sum_{k=2}^{D} (50(D+1) - 50k)$

→ next page

3.c. Continued

$C = 5000 + 50D + \cancel{50D} + 50(D+1)(D-1) - \sum_{k=2}^{D} 50k$

$C = 5000 + 50D + 50(D+1)(D-1) - \sum_{k=1}^{D} 50k + 50$

$\sum_{k=1}^{D} k = \frac{D(D+1)}{2}$

$C = 5050 + 50D + 50(D+1)(D-1) - \frac{50D(D+1)}{2}$

$C = 5050 + 50D + 50D^2 - 50 - 25D^2 - 25D$

$C = 5000 + 25D + 25D^2$

$\cancel{\frac{dC}{dD} = 50D + 25 = 0}$ The average cost per day $A = \frac{C}{D}$

$\cancel{D = -\frac{1}{2}}$ $A = \frac{5000}{D} + 25 + 25D$

$\frac{dA}{dD} = 25 - \frac{5000}{D^2} = 0$

$25D^2 = 5000$

$D = \pm 10\sqrt{2}$ → Only choose positive root

$\frac{d^2A}{dD^2} = \frac{10000}{D^3}$

at $D = 10\sqrt{2}$

$\frac{d^2A}{dD^2} = \frac{1000}{\sqrt{2}} > 0$ ∴ Minimum

Of course $D \neq 10\sqrt{2}$ as not an integer.

The closest integer is $D = 14$

To be sure this is the ~~minimum~~ integer value of D such that A is minimum,

$\frac{dA}{dD} < 0$ for $D \leq 14$ and $\frac{dA}{dD} > 0$ for $D \geq 15$

so $D = 14$ and $D = 15$ are the minimums for A

$A(14) = \frac{5125}{7} < \frac{2200}{3} = A(14)$

~~Thus the~~ → next page

~~#~~ 3.c. Continued

Thus the manufacturer should order 70 units every 14 days giving an average daily cost of

$A = \frac{\$5125}{7} = \$732$ (3sf)

Outstanding Scholarship

Subject: Calculus

Standard: 93202

Total score: 30

Q	Score	Marker commentary
1	08	The candidate exhibited elegance in tackling proof-style questions by methodically presenting work with logical clarity in Q1c, showcasing both a keen understanding and a commitment to coherence. They demonstrated consistency following through each step until the completion of the task, displaying determination and perseverance.
2	07	The candidate demonstrated a comprehensive grasp of complex numbers by adeptly expressing them in exponential form in Q2c. They successfully navigated and solved problems with minimal descriptions, highlighting a proficiency in applying complex number concepts to unfamiliar scenarios.
3	07	The candidate effectively applied differentiation rules in finding the derivative of the inverse tangent function in 3bii. They demonstrated adeptness in ‘problem-solving’ skills by employing mathematical models to effectively solve real-life optimisation question (3c) while stating assumptions. They would have achieved a perfect score if they had used the related rates of change rule in Q3bi.
4	08	The candidate showed a clear understanding of improper integrals, they successfully applied limits around points where the integral is undefined in Q4b. They also showcased a clear understanding of the Fundamental Theorem of Calculus in solving the differential equation in Q4d.