Assessment Schedule – 2023

Scholarship Calculus (93202)

Evidence Statement

Q	Solution
ONE (a)(i)	Q has coordinates (–6,8) and R (10,0). QR: $\dfrac{y-8}{x-(-6)}=\dfrac{0-8}{10-(-6)}$ $16y-128=-8x-48$ $y=-\dfrac{1}{2}x+5$
(ii)	The hyperbola and the circle have a common axis of symmetry, $y = -x$. As such, the line $y = -x$ is a normal to both RT and PQ, hence RT||PQ.
(b)	Label the equations as shown: $4\log_2(8x^3)+\log_5(y^6)=17$ A $\log_2(x^5)+\log_5(y^2)=3$ B From A: $4\log_2(8x^3)+\log_5\left((y^2)^3\right)=17 \Rightarrow$ $4\log_2(8x^3)+3\log_5(y^2)=17 \Rightarrow$ $12+12\log_2 x+3\log_5(y^2)=17$ C From B: $5\log_2 x+\log_5(y^2)=3$ D $12+12\log_2 x+3\log_5(y^2)=17$ $5\log_2(x)+\log_5(y^2)=3$ $\Rightarrow$ The system is now: $12\log_2 x+3\log_5(y^2)=5$ E $5\log_2(x)+\log_5(y^2)=3$ F Make the substitution: $a=\log_2 x$ and $b=\log_5(y^2)$ The system can be written as: $12a+3b=5$ $5a+b=3$ Which has solutions: $a=\dfrac{4}{3}$ and $b=-\dfrac{11}{3}$ Substituting back: $\log_2 x=\dfrac{4}{3}\Rightarrow x=2^{\frac{4}{3}}$ $\log_5\left(y^2\right)=-\dfrac{11}{3}\Rightarrow y=\pm 5^{-\frac{11}{6}}$

(c)	Let the point A have the coordinates $(x_1, y_1,)$ and B (x_2, y_2). $G_{\text{OA}} = \frac{y_1 - 0}{x_1 - 0}$ and $G_{\text{OB}} = \frac{y_2 - 0}{x_2 - 0}$ Since OA $\perp$ OB $\frac{y_1}{x_1} \times \frac{y_2}{x_2} = -1$ $y_1 y_2 = -\frac{y_1^2}{k} \times \frac{y_2^2}{k} \Rightarrow y_1 y_2 = -k^2$ (since $x = \frac{y^2}{k}$) $y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_2)$ For $y = 0, -y_1 = \frac{y_2 - y_1}{\frac{1}{k}\left(y_2^2 - y_1^2\right)}(x - x_1)$ $-y_1 = \frac{k}{y_2 + y_1}(x - x_1)$ $x = \frac{-y_1 y_2 - y_1^2}{k} + x_1$ $x = k - \frac{y_1^2}{k} + x_1 = k$ The x-intercept is $(k, 0)$. **Alternate solution :** Let A $= (a, b) \Rightarrow k = \frac{b^2}{a}$ grad OA $= \frac{b}{a} \Rightarrow$ grad OB $= -\frac{a}{b}$ OB $\Rightarrow y = \frac{-a}{b}x \Rightarrow x = \frac{-b}{a}y$ Solving for B $\quad y^2 = \frac{-kb}{a}y$ so B has coordinates $\left(\frac{kb^2}{a^2}, -\frac{kb}{a}\right)$ Now AB $\Rightarrow y - b = \frac{-\frac{kb}{a} - b}{\frac{kb^2}{a^2} - a}\left(x - a\right)$ When $y = 0$ $\frac{\frac{kb^2}{a^2} - a}{\frac{k + a}{a}} = x - a$ $\frac{\frac{k^2 - a^2}{a^2}}{\frac{k + a}{a}} = x - a$ $\frac{\left(k - a\right)\left(k + a\right)}{k + a} = x - a$ $x = k$

TWO (a)	Since $\frac{3\pi}{2} < x < 2\pi, \frac{3\pi}{4} < \frac{x}{2} < \pi$, so $\left(\frac{x}{2}\right)$ lies in the second quadrant and $\sin\left(\frac{x}{2}\right)$ is positive. Since $\sin x = \sqrt{\frac{1-\cos(2x)}{2}}, \sin\frac{x}{2} = \sqrt{\frac{1-\cos x}{2}}$ From $\cot x = -\frac{5}{12}$, we have $5, 12, 13$ triangle, and since x lies in the fourth quadrant, 12 is negative, which gives: $\cos x = \frac{5}{13}$ $\sin\left(\frac{x}{2}\right) = \sqrt{\frac{1-\frac{5}{13}}{2}} = \sqrt{\frac{4}{13}} = \frac{2}{\sqrt{13}}$
(b)	$(\overline{z})^4 = \left(-\frac{1}{2}-\frac{\sqrt{3}}{2}\mathrm{i}\right)^4 = \left[\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}\mathrm{i}\right)^2\right]^2$ $= \left[\frac{1}{4}-\frac{\sqrt{3}}{2}\mathrm{i}+\frac{3}{4}\mathrm{i}^2\right]^2 = \left(-\frac{1}{2}+\frac{\sqrt{3}}{2}\mathrm{i}\right)^2 = \frac{1}{4}-\frac{\sqrt{3}}{2}\mathrm{i}+\frac{3}{4}\mathrm{i}^2 = -\frac{1}{2}-\frac{\sqrt{3}}{2}\mathrm{i}$ Or $\overline{z} = \mathrm{cis}\frac{4\pi}{3}$ *and* $(\overline{z})^4 = \left(\mathrm{cis}\frac{4\pi}{3}\right)^4 = \mathrm{cis}\frac{16\pi}{3} = \mathrm{cis}\frac{4\pi}{3} = -\frac{1}{2}-\frac{\sqrt{3}}{2}\mathrm{i}$
(c)(i)	Since $\mathrm{i} = \mathrm{e}^{\mathrm{i}\frac{\pi}{2}}$, we have $\left(\left(\mathrm{e}^{\mathrm{i}\frac{\pi}{2}}\right)^{\mathrm{i}}\right)^2 = \left(\mathrm{e}^{-\frac{\pi}{2}}\right)^2 = \mathrm{e}^{-\pi}$ which is real.
(ii)	$\ln\left(-25\mathrm{e}^{\mathrm{i}^{\mathrm{i}}}\right) = \ln(-1) + \ln 25 + \mathrm{i}^{\mathrm{i}} \ln \mathrm{e}$ Now $-1 = \mathrm{e}^{\mathrm{i}\pi}$ so we have $\ln \mathrm{e}^{\mathrm{i}\pi} + \ln 25 + \mathrm{e}^{-\frac{\pi}{2}} = \left(\ln 25 + \mathrm{e}^{-\frac{\pi}{2}}\right) + i\pi$
(d)	$x^2 + y^2 = z\overline{z}$ $x = \frac{z+\overline{z}}{2}$ $y = \frac{z-\overline{z}}{2\mathrm{i}}$ Substituting into the xy plane circle: $Az\overline{z} + B\left(\frac{z+\overline{z}}{2}\right) + C\left(\frac{z-\overline{z}}{2\mathrm{i}}\right) + D = 0$ and $Az\overline{z} + \left(\frac{B}{2}+\frac{C}{2\mathrm{i}}\right)z + \left(\frac{B}{2}-\frac{C}{2\mathrm{i}}\right)\overline{z} + D = 0$ Let $\alpha = A, \gamma = D$, and $\beta = \frac{B}{2}+\frac{C}{2\mathrm{i}}$

THREE (a)	$x = 2t - 3t^3$ $y = te^t$ $\frac{dx}{dt} = 2 - 9t^2$ and $\frac{dy}{dt} = e^t + te^t$, so $\frac{dy}{dx} = \frac{e^t + te^t}{2 - 9t^2}$. Turning points when $\frac{dy}{dx} = 0$ Let $\frac{e^t + te^t}{2 - 9t^2} = \frac{e^t(1+t)}{2-9t^2} = 0 \Rightarrow t = -1$ $\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{e^t + te^t}{2 - 9t^2}\right) \times \frac{dt}{dx}$ $\frac{d}{dt}\left(\frac{e^t + te^t}{2 - 9t^2}\right) = \frac{\left[e^t + e^t + te^t\right]\left(2 - 9t^2\right) - \left[e^t + te^t\right]\left(-18t\right)}{\left(2 - 9t^2\right)^2}$ so $\frac{d^2y}{dx^2} = \frac{\left[e^t + e^t + te^t\right]\left(2 - 9t^2\right) - \left[e^t + te^t\right]\left(-18t\right)}{\left(2 - 9t^2\right)^2} \times \frac{1}{2 - 9t^2} = \frac{e^t\left[4 + 20t - 18t^2 - 9t^3\right]}{\left(2 - 9t^2\right)^3}$ and $\left.\frac{d^2y}{dx^2}\right\|_{t=-1} = \frac{e^{-1}\left[4 - 20 - 18 + 9\right]}{\left(2 - 9\right)^3} > 0$ So the curve has a minimum when $t = -1$, which gives the coordinates $\left(1, -e^{-1}\right)$
(b)(i)	Let the distance of the player from second base be $27.4 - x$, and the distance between the player and third base represented by l. Then $l^2 = 27.4^2 + (27.4 - x)^2$ Differentiating implicitly gives: $2l\frac{dl}{dt} = 2(27.4 - x)(-1)\frac{dx}{dt}$ When $x = 10$, $l = \sqrt{27.4^2 + 17.4^2} = 32.46$ m, so $\left.\frac{dl}{dt}\right\|_{x=10} = \frac{1}{2 \times 32.46}\left[2\left[27.4 - 10\right](-1) \times 5\right] = -2.68 \text{ m s}^{-1}$
(ii)	$\tan\theta_1 = \frac{27.4 - x}{27.4} = 1 - \frac{x}{27.4}$ and differentiating implicitly gives: $\sec^2\theta_1 \frac{d\theta_1}{dt} = \frac{-1}{27.4}\frac{dx}{dt} \Rightarrow \frac{d\theta_1}{dt} = \frac{-1}{27.4}\frac{dx}{dt} \times \cos^2\theta_1$ As the player touches base 2, $\theta_1 = 0$. So $\left.\frac{d\theta_1}{dt}\right\|_{x=27.4} = \frac{-1}{27.4}\frac{dx}{dt} \times \cos^2 0 = \frac{-1}{27.4}5 = -0.18 \text{ rad s}^{-1}$ And $\left.\frac{d\theta_2}{dt}\right\|_{x=27.4} = +0.18 \text{ rad s}^{-1}$

(c)	For a delivery every x days, $5x$ units must be ordered to have enough steel for that delivery cycle. The average amount in storage between deliveries is $\frac{5x-0}{2}=\frac{5x}{2}$. So, the cost of delivery and storage for each cycle is given by: Cost per cycle = delivery costs + storage costs, i.e Cost per cycle = $\$5000+\frac{5x}{2}\times x\times\$10$ For daily cost – divide through by number of days in the cycle. $c(x)=\frac{5000}{x}+25x$ where $x>0$. Since $\lim_{x\to 0}c(x)=+\infty$ and $\lim_{x\to+\infty}c(x)=+\infty$, a minimum must exist between these two limits. $\frac{\mathrm{d}c}{\mathrm{d}x}=-5000x^{-2}+25$ For minimum, set $0=-5000x^{-2}+25\Rightarrow x=\pm\sqrt{200}$ days. We use $\sqrt{200}$. $c\left(\sqrt{200}\right)=\frac{5000}{\sqrt{200}}+25\sqrt{200}=500\sqrt{2}\approx\$707.11$ Note $C''(x)=10000x^{-3}$ and $C''\left(\sqrt{200}\right)>0$, we do have a minimum. The gate maker should receive a delivery every $\sqrt{200}\approx 14$ days of $14\times 5=70$ units of steel. Other rounding arguments: $\sqrt{200}\times 5\approx 71$ A delivery every 14 days of 71 units. This will not incur any further charge for delivery since the transport costs are fixed. However, the extra 1 unit delivered will cost \$130 to store if used at the very end of the cycle. On the other hand, there is a manufacturing bottleneck of 5 units per day. Therefore a multiple of 5 is required.

FOUR (a)	Points of intersection of the two parabolae are: $6x - x^2 = x^2 - 2x \Rightarrow 2x^2 - 8x = 0 \Rightarrow x = 0 \text{ or } x = 4$ Intersection points: (0,0) and (4,8). $\text{Area} = \int_0^4 \left[\left(6x - x^2\right) - \left(x^2 - 2x\right)\right] \mathrm{d}x = \int_0^4 \left[8x - 2x^2\right] \mathrm{d}x$ $= \left[4x^2 - \frac{2}{3}x^3\right]_0^4 = 64 - \frac{128}{3} = \frac{64}{3} \text{ units}^2$
(b)	$\int_1^4 \frac{\mathrm{d}x}{(x-2)^{\frac{2}{3}}} = \int_1^2 \frac{\mathrm{d}x}{(x-2)^{\frac{2}{3}}} + \int_2^4 \frac{\mathrm{d}x}{(x-2)^{\frac{2}{3}}}$ $\int_1^2 \frac{\mathrm{d}x}{(x-2)^{\frac{2}{3}}} = \lim_{k \to 2^-} \int_1^k \frac{\mathrm{d}x}{(x-2)^{\frac{2}{3}}} = \lim_{k \to 2^-} \left[3(k-2)^{\frac{1}{3}} - 3(1-2)^{\frac{1}{3}}\right] = 3$ $\int_2^4 \frac{\mathrm{d}x}{(x-2)^{\frac{2}{3}}} = \lim_{k \to 2^+} \int_k^4 \frac{\mathrm{d}x}{(x-2)^{\frac{2}{3}}} = \lim_{k \to 2^+} \left[3(4-2)^{\frac{1}{3}} - 3(k-2)^{\frac{1}{3}}\right] = 3\sqrt[3]{2}$ Putting this together: $\int_1^4 \frac{\mathrm{d}x}{(x-2)^{\frac{2}{3}}} = 3 + 3\sqrt[3]{2}$
(c)	Noting the symmetry of the astroid and setting $t_1 = 0$ and $t_2 = \frac{\pi}{2}$, as well as using $\frac{\mathrm{d}y}{\mathrm{d}t} = 3\sin^2 t \cos t$ and $\frac{\mathrm{d}x}{\mathrm{d}x} = 3\cos^2 t(-\sin t)$, we have $L = 4\int_0^{\frac{\pi}{2}} \sqrt{\left(3\sin^2 t \cos t\right)^2 + \left(\left(3\cos^2 t\right)\left(-\sin t\right)\right)^2}\, \mathrm{d}t$ $L = 4\int_0^{\frac{\pi}{2}} \sqrt{9\sin^4 t \cos^2 t + 9\cos^4 t \sin^2 t}\, \mathrm{d}t$ $L = 4\int_0^{\frac{\pi}{2}} 3\sin t \cos t\, \mathrm{d}t$ $L = 12\int_0^{\frac{\pi}{2}} \cos t \sin t\, \mathrm{d}t$ $L = 6\left[\sin^2 t\right]_0^{\frac{\pi}{2}} = 6$ OR $L = 6\int_0^{\frac{\pi}{2}} \sin 2t\, \mathrm{d}t$ $L = 3\left[-\cos 2t\right]_0^{\frac{\pi}{2}} = 3\left(1 - -1\right) = 6$

(d)	Squaring both sides gives: $\left(f(x)\right)^2 = \int_0^x \left(f^2(t)+f'(t)^2\right)\mathrm{d}t + 2023$ then $\frac{\mathrm{d}}{\mathrm{d}x}\left(f(x)\right)^2 = \frac{\mathrm{d}}{\mathrm{d}x}\left(\int_0^x \left(f(t)^2+f'(t)^2\right)\mathrm{d}t + 2023\right)$ $2f(x)f'(x) = f(x)^2 + f'(x)^2$ $f(x)^2 - 2f(x)f'(x) + f'(x)^2 = 0$ $\left(f(x)-f'(x)\right)^2 = 0$ which gives $f(x) = f'(x)$ which has the solution $f(x) = C\mathrm{e}^x$ Substituting into the original equation gives: $C\mathrm{e}^x = \sqrt{\int_0^x \left((C\mathrm{e}^t)^2 + (C\mathrm{e}^t)^2\right)\mathrm{d}t + 2023}$ $C\mathrm{e}^x = \sqrt{C^2\int_0^x 2\mathrm{e}^{2t}\mathrm{d}t + 2023}$ $C\mathrm{e}^x = \sqrt{C^2\mathrm{e}^{2x} - C^2 + 2023}$ Squaring both sides gives: $C^2\mathrm{e}^{2x} = C^2\mathrm{e}^{2x} - C^2 + 2023$ and $C^2 = 2023 \Rightarrow C = \sqrt{2023}$ So the required function is $f\left(x\right) = \sqrt{2023}\,\mathrm{e}^x$ Alternatively: Substituting $x = 0$ into $C\mathrm{e}^x = \sqrt{C^2\mathrm{e}^{2x} - C^2 + 2023}$ gives $C = \sqrt{C^2 - C^2 + 2023} = \sqrt{2023}$

Sufficiency Statement

Score 1–4, no award	Score 5–6, Scholarship	Score 7–8, Oustanding Scholarship
Shows understanding of relevant mathematical concepts, and some progress towards solutions to problems.	Application of high-level mathematical knowledge and skills, leading to partial solutions to complex problems.	Application of high-level mathematical knowledge and skills, perception, and insight / convincing communication shown in finding correct solutions to complex problems.

Cut Scores

Scholarship	Outstanding Scholarship
17 – 28	29 – 32